Maa's formula for the modal density of a room is known to everyone in acoustics. The wonderful part of it is that it so logical and direct that you don't have to memorize or look it up - you can simply re-derive it on the spot. When it was published by Maa in 1939, there was a competing formulation by Dick Bolt that had the same volume dependent term, but was quite different in the surface area and edge length terms. But by the time of the landmark paper by Morse and Bolt in 1944 ("Sound Waves in Rooms", Rev Mod Phys vol 16, No. 1 April 1944), Maa's formula had won and was the relationship cited by Morse and Bolt. The uses of modal density in the response statistics of acoustical spaces and structures for the purposes of estimating impedances, energy flow, and phase statistics have grown as recognition of this fundamental property of resonant systems has grown.
I. Background

In the January 1939 issue of JASA, two consecutive papers on the mode count in rectangular rooms appeared [1,2], one by Dick Bolt, pp 228-234, and one by D-Y Maa, pp 235-238. The authors acknowledged one another, Bolt referring to Maa as “a graduate student in acoustics at the University of California...[1]” [not quite accurate, in 1939 Maa was a graduate student at Harvard working with F. V. Hunt]. The formulas for the mode counts were derived somewhat differently; the term for the volume dependence were the same, but the surface dependent terms differed. Maa states that “These equations agree very closely... [2]” [Not quite true, the area correction terms differ by the order of 20%]. In any case, Maa’s derivation of mode count was by far more direct and intellectually appealing than that of Bolt.

In 1941, a paper appeared in JASA by Glenn Roe, a graduate student in Physics at the University of Minnesota [3] that derived the mode count that included the volume and surface area terms. The spaces considered were spheres, half spheres, cylinders, half cylinders, and prisms. The formulas for these terms agreed with the derivation by Maa. When Phil Morse and Dick Bolt published their landmark paper “Sound Waves in Rooms” [4] in the April 1944 issue of the Reviews of Modern Physics, Bolt, Maa, and Roe were cited for their work, but the formulas for mode count presented were those of Maa.

2. Modal density for sound fields and structures

An important parameter in structural acoustics, closely related to mode count, is modal overlap. Conventionally, it is the average number of system modes excited to resonance by a compact, pure tone source. In rooms, the modal overlap is usually large, but in structures it can be small. When the modal overlap is large (greater than 10 for example), the statistics of system response (input power, mean square vibration amplitude, etc.) become simpler, allowing one to predict average and variance in
response. The well known “Schroeder frequency” for rooms occurs when the modal overlap is about

M=10. [5]

Modal overlap M=10 occurs at a relatively low frequency in a large room and the simplified statistics
apply over a large frequency region of interest. But structures generally have low modal overlap, which
introduces an additional factor, the modal spacing. The average modal spacing is of course the
reciprocal of the modal density, but the average spacing can result from a variety of spacing
distributions. A convenient model for the spacing between modes is a Poisson process that is used to
represent the spacing between noise events in a random signal. This process predicts a maximum in the
spacing probability at zero spacing. [6]

Schroeder studied the spacing of modes in a microwave cavity and found that Poisson statistics did not
hold – there was a deficiency in mode separations at small values, which he assigned to an inability to
resolve individual modes when they get too close together and they are therefore not counted. [7]

However, other studies of the energy levels in heavy nuclei, which also show the small spacing
deficiency, associate that deficiency with level repulsion, the tendency for resonators with nearly the
same resonance frequency to split when coupled. [7] The statistical model for level (or resonance
frequency) spacing is the “next neighbor” process. This process is produced by erasing every other event
in a Poisson process. [7, 8]

Detailed analysis of structural transfer functions (TF’s) using both the Poisson and Next Neighbor spacing
statistics show that the main effect of the spacing model is in the variance in the response amplitude at
small modal overlap. At high modal overlap, the spacing statistics don’t seem to matter. [9]
3. Log magnitude and phase of transfer functions

TF’s are usually expressed in dB for magnitude and radians or degrees for phase. The statistics of these functions are also determined in large part by mode counts and modal overlap. The phase function and its statistics are calculated using a one-dimensional random walk model. [10] It turns out that the mean and variance of the phase are determined by the mode count through the modal overlap. Positive and negative steps in phase of π radians occur as the poles and zeros of the TF are encountered as the frequency is increased. This leads to a “reverberant phase” determined by mode count somewhat analogous to the reverberant pressure in a reverberant room.

The magnitude (in dB) statistics are likewise determined by mode count and damping through modal overlap. Schroeder has shown that when his modal overlap criterion (M≥10) is met and the real and imaginary parts of the TF are gaussian distributed, the standard deviation of the TF is 6 dB. Since the overlap criterion is frequently not met for structures, a different approach shows that the standard deviation drops from Schroeder’s 6 dB to about 1 dB at low modal overlap.

4. Conclusion

The work of Maa, Bolt, Roe, Schroeder, and others on the calculation of mode count for a variety of acoustical spaces and structures has shown the importance of mode count in a variety of measures of response for a variety of resonant, modal systems. We are drawn to gratitude for their leading the way to help us understand the representation and analysis of response in complicated systems.
References


9. Reference 8, Section V.