4aAAa8. Time-domain formulation of an edge source integral equation

Peter Svensson* and Andreas Asheim

*Corresponding author’s address: Electronics and Telecommunications, Norwegian University of Science and Technology, O.S. Bragstads pl. 2B, Trondheim, NO-7491, -, Norway, svensson@iet.ntnu.no

In computer simulations of sound in enclosures, diffraction components can be added to geometrical acoustics ones for increased accuracy. A computational problem with diffraction is the large number of higher-order terms that is generated. A recent frequency-domain edge source integral equation (ESIE) efficiently handles the sum of all higher-order diffraction for rigid, external scattering objects, while computing first-order diffraction separately. Here, a time-domain formulation of the same ESIE is presented. An initial version handles higher-order diffraction for separate scattering objects, such as stage ceiling reflectors, and the extension to general geometries is outlined. With this approach, in a first step an incident transient sound field is computed at discretized edge points, including the outgoing directivity. In a second step, the effects of diffraction of arbitrarily high order is handled by solving the IE iteratively, yielding the complete edge source time signals. In a third step, the edge source signals are propagated to receiver points. Numerical issues will be discussed, including discretization strategies and how to handle shadow zone boundary singularities.

Published by the Acoustical Society of America through the American Institute of Physics
INTRODUCTION

The computation of scattering, and wave propagation in complex geometries and for high frequencies, is often done with geometrical acoustics (GA) based methods. These methods have superior efficiency compared to reference methods such as the finite element or finite differences method, or boundary element method. The accuracy of GA based methods is increased by the addition of edge diffraction components, and high accuracy has been demonstrated for rigid boundary conditions, both via comparisons with reference solutions and measurements. However, the addition of higher order diffraction components is challenging. Previous attempts to handle higher orders of diffraction have employed cascaded first-order diffraction expression with a discretization of each involved edge [1], [2], [3]. This implies a computational load which increases exponentially with diffraction order and it is costly both because of an increasing number of diffraction path combinations, and because of an increasing cost per component. Recently, an edge source integral equation (ESIE) was suggested as an alternative which permits the computation of arbitrarily high orders of diffraction at little additional computational cost. A frequency-domain (FD) formulation was presented in [4], and in this paper, a corresponding time-domain formulation is presented. The FD formulation was presented for rigid, convex scattering bodies, for which excellent accuracy was found for all frequencies. It is still uncertain how accurate the same approach is for non-convex bodies, since it has been pointed out that so-called slope diffraction needs to be handled for non-convex geometries [5]. However, the inaccuracy seems to be largest for low frequencies, and the worst-case geometry is a hole in a thin screen, where the ESIE incorrectly predicts zero higher-order diffraction. Still, in this worst case scenario, the maximum inaccuracy seems to be on the order of 2 dB. Another open question is how to handle non-rigid boundary conditions for which further work is needed. This paper focuses on scattering from bodies with rigid boundary conditions.

THE EDGE SOURCE INTEGRAL EQUATION

A frequency-domain formulation of the edge source integral equation (ESIE) has been presented earlier, in [4]. The relation between a time-domain (TD) and a frequency-domain (FD) formulation has been demonstrated before for first-order diffraction [6]. This relationship is quite straightforward, since the involved frequency dependence is of the form $e^{-jk\cdot r}$, which translates directly to a Dirac pulse in the time-domain.

Time-Domain Formulation of the Edge Source Integral Equation

The sound field will be modeled here as a time-domain formulation of a sum of geometrical acoustics (GA) components and edge diffraction components,

$$p_{total}(t) = p_{direct}(t) + p_{specular}(t) + p_{D1}(t) + \sum_{i=2}^{\infty} p_{Di}(t),$$  \hspace{1cm} (1)

where subscript $Di$ indicates diffraction components of order $i$. The GA components include the direct sound and specular reflections, and all the components in Eq. (1) involve visibility factors, i.e., only when there is a free line of sight between a source/receiver and the reflection/diffraction point on a surface, or between successive reflection/diffraction points, will a component be included. A monopole is assumed as an ensonifying source throughout. The GA components can be computed with, e.g., the image source method or beam tracing, whereas the first-order diffraction term is a sum of contributions from edge portions that are visible to both the source and the receiver. In addition, first-order diffraction might occur in sequences with preceding specular reflection, for which the visibility for the image source rather than the original source
FIGURE 1: A single wedge.

is to be used. Likewise, for sequences of specular reflections after an edge diffraction, the visibility for the image receiver should be used. By letting $S$ represent the original source or an image source, and $R$ represent the receiver or an image receiver, this can be summarized as

$$p_{D1}(t) = \int_{\Gamma} \delta \left( t - \frac{r_{R,z} + r_{z,S}}{c} \right) \frac{V_{R,z}V_{z,S}}{r_{R,z}r_{z,S}} \Omega(R,z,S) ds_z, \quad (2)$$

where the totality of edges in the polygonal model is represented by $\Gamma$, which could also be curved edges of thin discs. $r_{z_1,z_2}$ denotes the Euclidean distance between the points $z_1$ and $z_2$. It can be noted that a Dirac pulse is assumed as a monopole source signal which implies that the quantity $p(t)$ here is actually an impulse response. Furthermore, $V_{a,b}$ is a visibility factor which is 1 if point $b$ can be seen from point $a$, 0 if it can not be seen, 1/2 if the line from $a$ to $b$ exactly hits an edge, and a fraction of 1 when this line hits a corner. $\Omega$ is a directivity function for the path from source $S$, via edge point $z$, to the receiver $R$, given as

$$\Omega(R,z,S) = -\frac{\nu}{4\pi} \sum_{i=1}^{4} \frac{\sin(\nu \phi_i)}{\cosh(\nu \eta) - \cos(\nu \phi_i)}, \quad (3)$$

where $\nu$ is the wedge index, $\nu = \pi/\theta_W$, and the angles $\phi_i$ are

$$\phi_1 = \pi + \theta_S + \theta_R, \quad \phi_2 = \pi - \theta_S + \theta_R, \quad \phi_3 = \pi - \theta_S - \theta_R, \quad \phi_4 = \pi + \theta_S - \theta_R. \quad (4)$$

The quantity $\eta$ is given as

$$\eta = \cosh^{-1} \left( \frac{\cos \phi_S \cos \phi_R + 1}{\sin \phi_S \sin \phi_R} \right). \quad (5)$$

The angles $\phi_S$ and $\phi_R$ refer to the angle between the edge (or edge tangent) and $S$ and $R$, respectively, and the angles $\theta_S$, and $\theta_R$ refer to the angle between the plane and $S$ and $R$, respectively. This is illustrated in Figure 1.

The expression for first-order diffraction above have been given in many papers earlier, see e.g. [7], [1], albeit with a $\beta$ directivity function rather than an $\Omega$ as introduced here. Expressions for higher-order diffraction components have also been presented earlier, [1], but here a recently
developed frequency-domain integral equation formulation will be used as a basis for a
time-domain expression. Consequently, the term \( \sum_{i=2}^{\infty} p_{D_i}(t) \) in Eq. (1) is formulated as being
caused by an effective edge source signal, \( q(z_1, z_2, t) \) which accumulates all higher orders of
diffraction,

\[
p_{\text{HOD}}(t) = \sum_{i=2}^{\infty} p_{D_i}(t) = \int_{t_1}^{t_2} \int_{t_2}^{t_1} q(z_1, z_2, t - \frac{r_{R_1} + r_{z_1}}{c}) b_{z_1, z_2} V_{R_1, z_1} V_{z_1, z_2} \Omega(R, z_1, z_2) dz_2 dz_1, \quad (6)
\]

where \( b_{z_1, z_2} \) is 1/2 when the two edge points \( z_1 \) and \( z_2 \) are on the edges of the same polygonal
face, and 1 otherwise.

The effective edge source signal, \( q \), is found via a time-domain edge source integral equation
(TD-ESIE),

\[
q(z_1, z_2, t) = q_0(z_1, z_2, t) + \int_{t}^{t} q(z_2, z, t - \frac{r_{z_2}}{c}) V_{z_2, z} \Omega(z_1, z_2, z) dz,
\]
where \( q_0 \) represents the component caused by first-order diffraction from the original source \( S \).
A more compact version will be used for a matrix equation formulation, introducing a transfer
function, \( h(z_1, z_2, z) \), from point \( z \), via edge point \( z_2 \), to point \( z_1 \),

\[
h(z_1, z_2, z) = \frac{V_{z_2, z}}{r_{z_2}} \Omega(z_1, z_2, z),
\]

such that

\[
q(z_1, z_2, t) = q_0(z_1, z_2, t) + \int_{t}^{t} q(z_2, z, t - \frac{r_{z_2}}{c}) h(z_1, z_2, z) dz,
\]
and this is the TD-ESIE to solve.

**Solution Approach**

The Nyström method can be used for solving the integral equation. This simply means a
spatial discretization of Eq. (9) following some quadrature scheme,

\[
q(z_1, z_2, t) \approx q_0(z_1, z_2, t) + \sum_{i} w_i q(z_2, z_i, t - \frac{r_{z_2}}{c}) h(z_1, z_2, z_i),
\]

where \( w_i \) are weight functions corresponding to the discrete sample points \( z_i \). It is assumed here
that the geometry is described as a set of polyhedra and thin discs, which includes a set of edge
segments, straight and curved, and each segment will be sampled. The numbering will, however,
be such that \( i \) runs from 1 to the total number of sample points and subsets of \( i \) will belong to
each edge segment \( j \). As the simplest example of discretization scheme, uniform sampling would
imply that each edge segment, \( j \), of length \( l_j \) would have \( n_j \) sample points equally distributed,
and then \( w_i = l_j/n_j \) for the subset of sample points \( z_i \) that belong to edge segment \( j \). Superior
convergence will typically result from using Gauss-Legendre quadrature or even more
sophisticated techniques that employ the known singularity properties of the integrand.

The continuous-time (CT) expressions above need to be converted to discrete-time versions
in order to formulate a matrix equation for the TD-ESIE. A standard approach is to let the
discrete-time (DT) version, \( \tilde{x}(n) \), of a quantity \( x(t) \), be computed as the integral of the CT version
around each sample instant,

\[
\tilde{x}(n) = \int_{\frac{nT}{2}}^{\frac{(n+1)T}{2}} x(t) dt,
\]

where \( f_s \) is the sampling frequency. This integration is equivalent to convolving the CT signal
with a rectangular window, of one sample’s width, before sampling the signal, which


Page 4
sinc–function. This is a crude low-pass filter which explains why quite high oversampling ratios might be needed for high accuracy. More sophisticated ways are possible, based on convolving the CT signal with a more efficient anti-alias filter impulse response. Here, however, the simple filtering in Eq. (11) will be used throughout. Using an intermediate step where DT and CT signals are mixed, the discretized TD-ESIE can be written as

\[
\tilde{q}(z_1, z_2, n) \approx \tilde{q}_0(z_1, z_2, n) + \sum_{i} w_i q \left( z_2, z_i^i, t - \frac{r_{2z, z_i^i}}{c} \right) h(z_1, z_2, z_i^i) dt
\]

\[
= \tilde{q}_0(z_1, z_2, n) + \sum_{i} w_i h(z_1, z_2, z_i^i) \int_{n-0.5}^{n+0.5} q \left( z_2, z_i^i, t - \frac{r_{2z, z_i^i}}{c} \right) dt.
\] (12)

The last integral integrates the CT signal \( q \) over a short time interval, but it is the DT signal \( \tilde{q} \), and not the CT signal \( q \), which will be available. However, a CT signal can be reconstructed from its DT representation, as

\[
q(t) \approx f_S \sum_n w_r(t f_S - n) \tilde{q}(n),
\]

where \( w_r \) is a reconstruction function. The ideal one (whereby the approximation sign is replaced by an equals sign) is the sinc–function but here, the simple boxcar (rectangular window) reconstruction function will be used instead,

\[
w_r(x) = \begin{cases} 1, & |x| \leq 0.5 \\ 0, & |x| > 0.5, \end{cases}
\]

which leads to that

\[
q(t) \approx f_S \tilde{q}(\lfloor f_S \cdot t + 0.5 \rfloor).
\]

Therefore, the integral over a short time interval in Eq. (12) is

\[
\int_{n-0.5}^{n+0.5} q \left( z_2, z_i^i, t - \frac{r_{2z, z_i^i}}{c} \right) dt \approx f_S \tilde{q} \int_{n-0.5}^{n+0.5} q \left( z_2, z_i^i, f_S \cdot \left( t - \frac{r_{2z, z_i^i}}{c} \right) + 0.5 \right) dt
\]

\[
= a \tilde{q} \left( z_2, z_i^i, f_S \cdot \left( n - \frac{0.5}{f_S} - \frac{r_{2z, z_i^i}}{c} \right) + 0.5 \right) + (1-a) \tilde{q} \left( z_2, z_i^i, f_S \cdot \left( n + \frac{0.5}{f_S} - \frac{r_{2z, z_i^i}}{c} \right) + 0.5 \right)
\]

\[
= a \tilde{q} \left( z_2, z_i^i, n - \frac{f_S r_{2z, z_i^i}}{c} \right) + (1-a) \tilde{q} \left( z_2, z_i^i, n - \frac{f_S r_{2z, z_i^i}}{c} \right) + 1,
\] (13)

where

\[
a = a \left( n - \frac{f_S r_{2z, z_i^i}}{c} - \lfloor n - \frac{f_S r_{2z, z_i^i}}{c} \rfloor \right) = a \left( \frac{f_S r_{2z, z_i^i}}{c} - \lfloor \frac{f_S r_{2z, z_i^i}}{c} \rfloor \right),
\]

\[
a(x) = \begin{cases} 1 - |x|, & x < 1 \\ 0, & |x| > 1. \end{cases}
\]

Accordingly the DT ESIE, on the Nyström form in Eq. (12), can be written as a convolution sum

\[
\tilde{q}(z_1, z_2, n) \approx \tilde{q}_0(z_1, z_2, n) + \sum_i w_i h(z_1, z_2, z_i^i) \sum_{m=0}^{\min(n, N_{h, max})} a \left( n - m - \frac{f_S \cdot r_{2z, z_i^i}}{c} \right) \tilde{q} \left( z_2, z_i^i, n - m \right).
\] (14)

The upper limit of the summation is the lowest of \( n \) and \( N_{h, max} \). The former is because the system is assumed to be at rest before the time sample \( n = 0 \). The latter condition is given by the longest possible delay within the system, between the two edge points that are the furthest apart.

After introducing the discretized versions of all quantities, each pair of edge sample points, \((z_1^i, z_2^i)\), that can see each other will generate one unknown signal, \( \tilde{q}(z_1^i, z_2^i, n) \), and all of these
\( \tilde{q}(z_1, z_2, n) \) can be stacked in one large matrix \( \mathbf{q} \). Now, the TD-ESIE in Eq. (14) will need to be solved using a time-marching approach. Thus, one column of \( \mathbf{q} \) will have to be solved for at a time, starting with the first column corresponding to \( n = 0 \) etc. One such column will be denoted \( \mathbf{q}_n \) and then

\[
\mathbf{q}_n = \mathbf{q}_{0,n} + (\mathbf{h} \circ \mathbf{a}_0)(\mathbf{w} \circ \mathbf{q}_n) + (\mathbf{h} \circ \mathbf{a}_1)(\mathbf{w} \circ \mathbf{q}_{n-1}) + \ldots
\]  

(15)

where \( \circ \) denotes the Hadamard product, i.e., an element-wise product (in Matlab \( \ast \)). Note that the matrix \( \mathbf{a}_0 \) has non-zero values only for those edge-to-edge combinations that reach each other within the first sample slot. This is a very small fraction of all the edge-to-edge combinations and therefore the \( \mathbf{h} \circ \mathbf{a}_0 \) matrix is a “sifted” version of \( \mathbf{h} \). In total, the \( \mathbf{h} \) matrix is spread out across a number of such sifted versions, and each (non-zero) value of \( \mathbf{h} \) appears only in two of the sifted matrices, because Eq. (13) shows that each edge-to-edge combination is spread out across only two time samples. Eq. (15) will contain \( N_{h,\text{max}} \) terms. One important detail relates to the sampling frequency vs. the shortest distance between two edge points. The sampling frequency can be chosen high enough that \( \mathbf{a}_0 \) is zero, and then the time-marching in Eq. (15) is straightforward. However, if a lower sampling frequency is chosen, then the RH side in Eq. (15) will contribute also to \( \mathbf{q}_n \), such that a rewriting into a matrix inversion form will be necessary. In the numerical examples below, high enough sampling frequencies are chosen in order to avoid the involved matrix inversion.

In [4] it was pointed out that the FD ESIE could be solved either directly, via a matrix inversion, or using iteration whereby each iteration step corresponded exactly to one diffraction order. For the TD ESIE formulation presented here, however, the time-marching approach will not separate diffraction orders. Rather, the complete solution will result, sample by sample. Since the response will be infinitely long, a truncation at a suitably low amplitude will have to be employed. It can be noted that this time-marching approach appears not to possess any instability tendencies for the tested sampling frequencies.

Previous approaches to second-order diffraction, and beyond, have constructed each complete path from source to edge point (possibly via specular reflections), via \( n \)-th order diffraction to another edge point, and finally to the receiver (possibly via yet another sequence of specular reflections). The crucial difference here is that the all paths are constructed from the source to an edge point (possibly via specular reflections), but then stored at that point. For each edge point, however, the amplitude in the direction of each visible other edge point is computed. After that, all possible edge-to-edge transfers are computed using iteration, ending with yet another set of amplitudes for each edge point, in the direction of each visible other edge point. As a final step, these amplitudes, called edge source signals, are propagated to the receiver points (possibly via a sequence of specular reflections). This approach indicates that all the visible edge-to-edge paths must be known when the first source term \( \mathbf{q}_0 \) is computed. Furthermore, the inclusion of specular reflections in-between edge diffraction events requires additional pre-processing before the source term: all the relevant specular reflections must be established. Apparently, this is a huge complication but in this study, only convex geometries are studied. This focus on convex geometries follows the method in [4] where it was acknowledged that non-convex geometries would require the introduction of so-called slope diffraction terms.

**Numerical examples**

Numerical results will be presented here for one specific case: the scattering from a circular rigid disc for an on-axis incident wave. A six-sided polygonal approximation of a disc is used, a monopole source is placed at a height of 10 radii, and a receiver is placed centrally on the disc, right at the surface of the source-side of the disc, see Fig. 2.
Comparison between the frequency- and the time-domain ESIE results

The TD ESIE, as given by Eq. (15), and the following propagation integral in Eq. (6), were implemented in Matlab. Each edge was discretized with 8 or 16 points, following a uniform or a Gauss-Legendre weighting scheme. A sampling frequency of 55 kHz was chosen, which gave a minimum of one sample’s delay between all edge point pairs. Sufficiently long time histories were computed such that the edge source signals had decayed to insignificant levels. As reference solution, a FD computation was carried out with a 32-point Gauss-Legendre discretization of each edge, following the method in [4]. The TD results were converted to the FD using FFT. Gauss-Legendre quadrature was used for all the results in Fig. 3.

Two types of errors will affect the results. One is caused by the spatial discretization, which is identical for the TD and the FD method. This type of error initially grows relatively slowly with frequency, up to a point where there are too many oscillations over the integration range, and the error increases rapidly. Around $ka = 25$, where the error for the “FD 8 GL” case starts to increase, and correspondingly around $ka = 50$, for the “FD 16 GL” case, the discretization then gives 1.8 points per $\lambda$ (on average; since the GL points are distributed non-uniformly). A second source is the time-discretization which obviously occurs only for the TD results. The simple time-discretization scheme which is employed here is expected to give an error which increases as +6 dB per octave for this case. These two error mechanisms can be observed in the results. At very low frequencies, the TD error is negligible, and the TD and the FD formulations give identical accuracy. At some frequency, the increasing TD error starts to dominate. Then, at high enough frequencies, either of the two mechanisms might be dominating. A complementary view is offered by the results in Fig. 3b, where the results with 16 GL points in Fig. 3a are repeated, plus one additional curve. The added results are computed with the TD method, using the same spatial discretization, but with a 4 times oversampling. This oversampling (a sampling frequency of 220 kHz) then leads to an 80-times oversampling at the frequency $ka = 25$, which is the highest that is presented here. For many applications, an accuracy of $10^{-3}$ is not required and a much lower oversampling ration can be accepted. One of the attractive features of the TD formulation is that it provides results for many frequencies at once. It is therefore highly desirable that the sought frequency range is a large fraction of the sampling frequency. Clearly, however, these results indicate that higher-order time discretization schemes are needed.

The impulse response function

One example of an impulse response computed with the TD-ESIE is presented in Fig. 4a. Also plotted are impulse responses computed with the order-by-order computation described in [1]. All impulse responses were computed with the same spatial discretization (16 Gauss-Legendre points), a sampling frequency of 220 kHz, and were subsequently lowpass filtered by convolving with a 99-sample wide rectangular window. This lowpass filtering reduces the amplitude by 6 dB around 1340 Hz, $ka = 24.5$. What appears as noise between 4 and 8 ms
FIGURE 3: Sound pressure amplitude, re. free-field, computed with the ESIE for the case in Fig. 2. The errors are absolute errors (difference from the reference solution). FD = frequency-domain results and TD = time-domain results. The frequency axis is given as $ka$ where $a$ is the equivalent radius for a circle of the same area as the six-sided disc.

(a) Comparison between the TD and FD formulations, and between 8 and 16 quadrature points. (b) Same results as in (a), with the addition of a curve with four times higher sampling frequency for the TD formulation.

are inaccuracies at high frequencies, caused by the limited number of quadrature points.

Computational load

The dominating computational part of the formulation above is the setting up of the $h \circ a_i$ matrices, and the iterative matrix multiplication and summation procedure, described by Eq. (15). In the same way as for the Boundary Element Method, the subsequent propagation of the sound field to a number of receiver positions, by Eq. (6), is typically a small part of the total computational load. The FD formulation of the ESIE has a computational complexity which is dominated by the number of elements in a transfer function matrix, which is $O(n_{edge\ points}^3)$ for each frequency. The TD formulation has the same number of non-zero elements in the large transfer matrix, but these elements are distributed across a number of very sparse $h \circ a_i$ matrices. The number of such matrices is determined by the number of unique integer propagation delays for the totality of edge-to-edge point set. When oversampling is used, this number of unique delays tends to grow more than linearly with the number of edge points. If oversampling is not used, then the number of unique sample delays would not change much with number of edge points. In the prototype Matlab implementation, where oversampling was employed, a complexity of $O(n_{edge\ points}^3)$ has been observed, but this number will be affected by how efficiently the set of sparse matrices can be set up and handled. Ultimately, a complexity closer to $O(n_{edge\ points}^3)$ could be targeted for the TD ESIE formulation. Since the number of edge points would be proportional to the highest frequency of interest, the complexity could potentially come close to $O(f_{max}^3)$.

CONCLUSIONS

A time-domain formulation of the edge source integral equation (ESIE) has been presented. The accuracy is comparable to frequency-domain results, with the additional error source caused by the time-discretization. It has been shown that a simple time-discretization scheme requires high oversampling ratios for high accuracy. The computational cost of the time-domain formulation has been analyzed for a prototype Matlab implementation. The clearly dominating
cost is the setting up of the transfer matrices, and the time-marching use of these matrices. A cost of $O(n^5_{\text{edge points}})$ was observed, and the high exponent was caused by the high oversampling used, which lead to too sparsely distributed transfer matrices.

**REFERENCES**


