Acoustic propagation modeled by the curvilinear Fourier pseudospectral time-domain method

Maarten Hornikx* and Daan Steeghs

*Corresponding author's address: Eindhoven University of Technology, Eindhoven, 5600 MB, Noord-Brabant, Netherlands, m.c.j.hornikx@tue.nl

The Fourier pseudospectral time-domain method is an efficient domain-discretization wave-based method to model sound propagation in inhomogeneous bounded media. The method was successfully applied to model atmospheric sound propagation and acoustics in urban environments. One of the limitations of the method is its restriction to a Cartesian grid, confining it to staircase-like geometries. When applying a transform from the Cartesian coordinate system to the curvilinear coordinate system, more arbitrary geometries may be solved by the method. In free field, the frequency dependent accuracy of the curvilinear Fourier pseudospectral time-domain method is investigated as a function of the deformation angle of the grid. Further, the performance of the curvilinear pseudospectral method is investigated for sound propagation in a box and for scattering from a rigid body. Finally, the sound field in a concert hall configuration with realistic boundary impedance values is computed. All computed results are in 2D and show good agreement against the boundary element method for low deformation angles.

Published by the Acoustical Society of America through the American Institute of Physics
INTRODUCTION

Partly driven by the advances in computer power, wave-based modeling for room acoustics has attracted increased attention in recent years. Wave-based modeling is especially of interest for treating problems where effects as diffraction are important, for auralization purposes as well as for sound propagation in inhomogeneous media. The latter being of interest mostly in outdoor acoustics. Along with the increase of computer power, developments within the numerical models to more efficiently solve the governing acoustic equations have taken place. Some important developments are efficient finite-difference time-domain (FDTD) schemes (Kowalczyk, 2011) and the application of the fast multipole boundary element method (FMBEM) to room acoustics (Atak, 2010). Another efficient time-domain method is the Fourier pseudospectral time-domain method (PSTD). PSTD is an attractive full volume-discretization method as only 2 spatial points are required to resolve a wavelength at spectral accuracy. It enables to solve the linearized Euler equations (LEE), and detailed sound propagation including reflection, diffraction and meteorological effects has accurately been predicted (Hornikx, 2010). A limitation is the modeling of boundary conditions, but treatment of rigid boundaries and boundaries with a different density have successfully been presented (Hornikx, 2010), and an approximation for impedance boundary conditions has been made (Spa, 2011). The current PSTD method can only accurately solve staircase-type geometries. For non staircase-type geometries that occur in real-life, e.g. inclined or curved surfaces, the method is only approximate. In related numerical methodologies to the Fourier PSTD method, i.e. the Chebyshev PSTD method and the FDTD method, a transform of the coordinate system has been employed in order to relax the constraints imposed by the Cartesian grid. An often-used approach is the curvilinear coordinate transform (Marsden, 2005; Zeng, 2004). Although application of the curvilinear coordinate transform for the Fourier PSTD method has appeared in earlier work (Nielsen, 1994), the combination with appropriate boundary conditions as presented in (Hornikx, 2010) has not been encountered. Here, it is investigated whether the curvilinear transform can successfully be applied to the Fourier PSTD method. The curvilinear transformation of the LEE is described in Section 2. Thereafter, the method for solving the equations is shortly addressed. The geometrical configurations as studied in this work are then presented, and the algorithm used to map an arbitrary geometry onto a Cartesian grid is described. The performance of the curvilinear PSTD method is studied for various configurations: free field, an interior problem and an exterior problem. Finally, an application to compute the sound field in a concert hall is carried out. All results involving boundaries are compared with results from boundary element calculations using the openBEM codes (Henriquez, 2010). All computations in this paper are performed in 2D.

METHODOLOGY

Curvilinear Linearized Euler Equations

Acoustic propagation through a medium at rest bounded by boundaries of arbitrary shape can be described by the homogeneous linearized Euler equations. In a 2D coordinate system with coordinates \((x,y)\), they read:

\[
\frac{\partial \textbf{q}}{\partial t} + \textbf{A}_j \frac{\partial \textbf{q}}{\partial j} = 0, \quad \textbf{A}_j = \begin{bmatrix} 0 & 0 & \delta_{x,j} \\ 0 & \frac{\rho_0}{\rho_0} & \delta_{z,j} \\ \rho_0 c_0 \delta_{x,j} & \rho_0 c_0 \delta_{z,j} & 0 \end{bmatrix},
\]

with the acoustic variable vector \(\textbf{q} = [u_x, u_z, p]^T\), with the velocity components \(u_x\) and \(u_z\) and the acoustic pressure \(p\). Further, \(\delta_{i,j}\) is the Kronecker delta, index \(j\) equals \(x\) or \(z\), \(\rho_0\) and \(c_0\) are the density and adiabatic sound speed of the propagation medium, air in this case. Together with boundary and initial conditions, the acoustic problem is described. A popular way to solve such a problem is by meshing the whole domain by a Cartesian mesh and to approximate the solution at every mesh point through numerically solving Eq. (1) by FDTD or PSTD. However, the Cartesian mesh will not capture arbitrarily shaped boundaries, i.e. non staircase-type boundaries, in an accurate way. To be able to utilize a solution method as FDTD and PSTD on a Cartesian mesh, a coordinate transform is applied from the physical domain \((x,y)\) to the Cartesian domain.
The Cartesian domain can thus be expressed as a function of the physical domain coordinates, i.e. \( \xi = \zeta(x, y) \) and \( \eta = \zeta(x, y) \). Equation (1) in Cartesian coordinates then reads

\[
\frac{\partial \mathbf{q}}{\partial t} + A_i \frac{\partial \mathbf{q}}{\partial x} = 0, \quad A_i = \begin{bmatrix}
0 & 0 & \xi_x \delta_{\eta,i} + \eta_x \delta_{\xi,i} \\
0 & 0 & \xi_x \delta_{\eta,i} + \eta_x \delta_{\xi,i} \\
\rho_0 c_0 \left( \xi_x \delta_{\xi,i} + \xi_y \delta_{\eta,i} \right) & \rho_0 c_0 \left( \eta_x \delta_{\xi,i} + \eta_y \delta_{\eta,i} \right) & 0
\end{bmatrix},
\]

(2)

with \( \xi_x = \frac{\partial \xi}{\partial x} \) etc., and \( l \) equals \( \xi \) or \( \eta \). The acoustic variable vector \( \mathbf{q} \) can thus be solved in the Cartesian coordinate system through solving Eq. (2).

**Pseudospectral time-domain method**

The numerical method to solve Eq. (2) as used in this paper, the Fourier pseudospectral time-domain (PSTD) method, is described in this Section. For a more rigorous presentation of the method the interested reader is referred to (Hornikx, 2010). The geometries of interest are discretized by an equidistant Cartesian mesh as depicted in Figs. 1-3(b). The equations are marched in time by a low-storage optimized 6-stage Runge-Kutta method (Bogey, 2004). At every discrete time step, the spatial derivatives are computed using the Fourier pseudospectral method. For evaluation of the \( \xi \)-derivative of pressure and a velocity component \( u_j \) (with \( j \) any direction) along an unbounded \( \xi \)-direction with dimension \( M \Delta \xi \), the following expressions are used:

\[
\begin{align*}
\frac{\partial p(n \Delta \xi^n)}{\partial \xi} &= \mathcal{F}_\xi^{-1} \left( jk \xi e^{-jk \frac{\Delta \xi}{2}} \mathcal{F}_\xi \left[ p(m \Delta \xi) \right] \right), \quad 0 \leq m \leq M - 1, \\
\frac{\partial u_j(n \Delta \xi^n)}{\partial \xi} &= \mathcal{F}_\xi^{-1} \left( jk \xi e^{jk \frac{\Delta \xi}{2}} \mathcal{F}_\xi \left[ u_j(m \Delta \xi^n) \right] \right), \quad 0 \leq m \leq M,
\end{align*}
\]

(3)

where \( \Delta \xi^n \) denotes \( \Delta \xi + \frac{\Delta \xi}{2} \), \( \mathcal{F}_\xi \) is the forward and \( \mathcal{F}_\xi^{-1} \) the inverse discrete Fourier transform over and to the \( \xi \)-variable, and \( 0 \leq n \leq M \). A spatial grid with staggered pressure and velocity nodes is used. To evaluate the derivatives at positions staggered by \( \pm \frac{\Delta \xi}{2} \), the spatial derivatives are multiplied in the wave number domain by \( e^{\pm jk \frac{\Delta \xi}{2}} \). Since FFTs are used to calculate derivatives, only two spatial points per (projected) wavelength are needed with this method. Derivatives along directions with a boundary medium with a different density then air are used here too as presented in (Hornikx, 2010). For the \( \xi \)-derivative of pressure and normal velocity component \( u_\xi \) along \( \xi \)-direction with a boundary interface at \( \xi = 0 \) and total dimension \( 2M \Delta \xi \), the following expressions are obtained

\[
\begin{align*}
\frac{\partial p(n \Delta \xi^n)}{\partial \xi} &= \mathcal{F}_\xi^{-1} \left( jk \xi e^{-jk \frac{\Delta \xi}{2}} \mathcal{F}_\xi \left[ p_1 \right] \right) \quad -M \leq n \leq -1, \\
&\quad 0 \leq n \leq M_1 - 1, \\
p_1 &= \begin{cases}
p(m \Delta \xi) & -M \leq m \leq -1, \\
\mathcal{R}_{1,1} p(-m \Delta \xi) + \mathcal{T}_{2,1} p(m \Delta \xi) & 0 \leq m \leq M - 1,
\end{cases} \\
p_2 &= \begin{cases}
p(m \Delta \xi) & 0 \leq m \leq M - 1, \\
\mathcal{R}_{2,2} p(-m \Delta \xi) + \mathcal{T}_{2,1} p(m \Delta \xi) & -M \leq m \leq -1,
\end{cases} \\
\frac{\partial u_\xi(n \Delta \xi^n)}{\partial \xi} &= \mathcal{F}_\xi^{-1} \left( jk \xi e^{jk \frac{\Delta \xi}{2}} \mathcal{F}_\xi \left[ u_\xi,1 \right] \right) \quad -M \leq m \leq -1, \\
&\quad 0 \leq m \leq M - 1, \\
u_\xi,1 &= \begin{cases}
u_\xi(m \Delta \xi^n) & -M \leq m \leq -1, \\
-\mathcal{R}_{1,1} u_\xi(-m \Delta \xi^n) + \mathcal{T}_{1,2} u_\xi(m \Delta \xi^n) & 0 \leq m \leq M - 1,
\end{cases} \\
u_\xi,2 &= \begin{cases}
u_\xi(m \Delta \xi^n) & -M \leq m \leq -1, \\
-\mathcal{R}_{2,2} u_\xi(-m \Delta \xi^n) + \mathcal{T}_{2,1} u_\xi(m \Delta \xi^n) & 0 \leq m \leq M - 1.
\end{cases}
\end{align*}
\]

(4)

with \( \mathcal{R}_{i,j} \) and \( \mathcal{T}_{i,j} \) the physical reflection and transmission coefficients from medium \( i \) to medium \( i \) or \( j \). The indices 1 and 2 correspond to the media at negative and positive locations from the interface. Non-reflective boundaries are treated by including an absorption layer at the edges of the computational domain by means of the perfectly matched layer (PML); see (Hornikx, 2010). Apart from the free field configuration, the curvilinear transform of Eq. (2) has been applied as a conformal map in this work. A property of conformal
mapping is that it preserves, from the Cartesian to the curvilinear grid, the grid angles between the grid lines. An advantage of this property near the boundaries is that grid lines are either parallel or perpendicular to the boundaries. This implies that Eqs. (4) to compute the pressure derivatives is appropriate. However, stretching of the physical grid relative to the Cartesian grid (see e.g. Fig. 2(a)) causes reduced wavelengths in the Cartesian coordinate system. The consequence is a lower maximum resolved frequency. Equations (4) to compute the normal velocity component will generally only be approximate for the computation of the \( \zeta \)-derivative of the \( u_x \) and \( u_z \) components (which are generally not following \( \zeta \)-direction). This will be a limitation in the used methodology. The boundary medium is approached as being locally reacting by not computing derivatives in the boundary medium, parallel to the interface. This approach is approximative too, as Eqs. (4) are strictly valid for media with equal speeds of sound in all directions.

**Mapping algorithm and configurations**

For the discretization of the physical configurations and the mapping of the physical mesh to the Cartesian mesh, two approaches are used. In the free field configuration (called C1), a deformed rectangle is modeled by a non-conformal curvilinear grid, see Fig. 1(a). The transform reads as:

\[
\xi = x + z \tan(\theta), \quad \eta = z,
\]

For the three configurations with boundaries, a Schwarz-Christoffel conformal map is made from the physical configuration to a rectangle, utilizing the Schwarz-Christoffel toolbox (Driscoll, 1996). For the interior problem (C2), a rectangular box with an inclined surface at one side is chosen, see Fig. 2(a). The angle of the inclined surface is changed to investigate the accuracy of the curvilinear PSTD method for this configuration. For the exterior scattering problem (C3), the left and right surface from a rigid rectangle are deformed, see Fig. 3. For this problem, the mesh representing the physical domain is composed of 4 subdomains: the domains left and right from the object are a conformal map and its mirror image. The domains above and below the object are untransformed grids. Finally, a concert hall shape is modeled (C4), see Figs. 4.

All calculations are initiated by an initial pressure distribution \( p(x,z) = e^{-b|x-x_s|} \), with \( x = (x,z) \) and \( x_s \) the source location and \( b = \frac{3^{-5}c_0^2}{\Delta^2} \). Here, \( \Delta \) is the uniform grid spacing in the Cartesian grid.

**Results**

Frequency domain results in this section are presented in two ways, either by the excess attenuation EA or by the averaged error \( \epsilon \). The receiver-position averaged error \( \epsilon \) for all 4 configurations (C1, C2, C3 and C4) is computed as:

\[
\epsilon = \frac{1}{N} \sum_{n=1}^{N} |\Delta L_n|\]

- **C1**: \( \Delta L = L_{p,PSTD}(\theta) - L_{p,PSTD}(\theta = 0^\circ) \)
- **C2,C3**: \( \Delta L = EA_{BEM}(\theta) - EA_{PSTD}(\theta) \)
  \[= (L_{p,BEM}(\theta) - L_{p,BEM}(\theta = 0^\circ)) - (L_{p,PSTD}(\theta) - L_{p,PSTD}(\theta = 0^\circ)) \]
- **C4**: \( \Delta L = EA_{BEM} - EA_{PSTD} \)
  \[= (L_{p,BEM} - L_{p,BEM,free}) - (L_{p,PSTD} - L_{p,PSTD,free}) \]

with \( L_p \) the sound pressure level. Error values \( \epsilon \) are presented for 1/3 octave band values. The levels are then first computed per 1/3 octave band, where the reference levels (i.e. for \( \theta = 0^\circ \) or the free field case) have been normalized to a flat spectrum.
The accuracy of the curvilinear PSTD method for the deformation of a Cartesian grid in free field as shown in Fig. 1(a) is evaluated first. The outer boundaries are untreated and thereby periodic boundary conditions. Only the direct part of the computed responses at the receivers is considered in this analysis. The results are compared to the solution for $\theta = 0^\circ$. Fig. 1(c) shows the waveform recorded at $\mathbf{x}_r = (2\cos(\pi/4), 2\sin(\pi/4))$ for $\theta = 45^\circ$, showing good agreement with the Cartesian solution of $\theta = 0^\circ$. The averaged error $\epsilon$ for all receiver points circumfering the source (72 in total), as depicted in Fig. 1(a), as a function of the grid deformation angle $\theta$ and grid resolution in terms of points per wavelength, is shown in Fig. 1(d). Results show that the number of grid points needed for accurate results increase with increasing deformation angle. Also, above $\theta = 50^\circ$, a finer discrete time step is needed to prevent the solution turning instable. The higher demand on the spatial resolution and discrete time step for larger deformation angles is caused by the increasing distance between grid points in $\eta$-direction in the physical grid, reducing the maximum resolved wavelength.

**Free field (C1)**
Figure 2: Interior configuration (C2) of a box with inclined left side. All boundaries have been modeled by a real-valued normalized impedance of $Z = 78$. The boundary media are locally reacting and explicitly modeled, but not shown in the figures. (a) Physical configuration with source at $x_s = (3, 0.5)$ and receivers at $x_r = (-0.2, z)$; (b) Cartesian configuration; (c) Excerpt of EA at $x_r = (-0.2, -4)$ for $\theta = 10^\circ$; (d) 1/3 octave band values of $\epsilon$ for various angles of $\theta$, see Eq. (6).

Interior configuration (C2)

For the interior configuration, all walls of the 10.4 m x 10 m box have a real-valued normalized impedance value of $Z = 78$, corresponding to an absorption coefficient of $\alpha_n = 0.05$ for normal incident sound waves. As reference calculations with the openBEM code (Henriquez, 2010) are in the frequency domain, this little absorption is necessary to get good agreement. As derivatives are not computed in the boundary medium parallel to the interface (as they are treated as locally reacting), the derivatives perpendicular to the interface of velocity components parallel to the interface in the propagation medium, are computed by treating the boundary are a pressure release boundary. The initial pressure distribution has its center at $x_s = (x, z) = (3, 0.5)$ and receivers are located at at $x_r = (-0.2, z)$, see Fig. 2(a). The excess attenuation EA relative to the box with $\theta = 0^\circ$ is computed both from BEM and from curvilinear PSTD results according to Eq. (6). The results for $\theta = 10^\circ$ are shown in Fig. 2(c) for an excerpt of the spectrum, showing good agreement. The error $\epsilon$ as shown in Fig. 2(d) is below 0.5 dB for angles up to $\theta = 15^\circ$ and for resolutions coarser than about 3 points per wavelength. The maximum resolved frequency is clearly shifted to a larger number with increasing angle $\theta$. For $\theta = 20^\circ$, a finer discretization ($\Delta/2$) and filter were needed to prevent instability. The accuracy is thus clearly more limited than for the free field configuration, caused by the approximate way of treating the boundaries.
As exterior problem, the scattering from a rigid deformed rectangle is studied, see Fig. 3(a), corresponding to a rigid rectangle in Cartesian coordinates. The outer boundary is modeled by a PML layer, absorbing the sound waves. Three angles $\theta$ are considered, corresponding to a maximum body width of 0.75 m, 1.0 m and 1.5 m. Figure 3(c) shows the $EA$ values relative the body with $\theta = 0^\circ$ for a single frequency as a function of the receiver angle for $\theta = 7.0^\circ$. The result shows a good agreement for lower angles, but a lower agreement for high angles, i.e. behind the body. For the three angles of $\theta$, the receiver position averaged error is shown in Fig. 3(d). The error increases for increasing value of $\theta$. In general, a larger error is found than for the interior problem. This is likely caused by the coupling of domains, with mapping derivative-terms as arise Eq. (2) being non-smooth across subdomain interfaces.
Concert hall configuration (C4)

To show the potential of the presented methodology, a concert hall shaped cross section is now modeled, see Fig. 4(a). As for configurations C2 and C3, the Cartesian domain of this configuration is a rectangle. A single source and receiver are located at $x_s = (-17,-6)$ and $x_r = (5,-2.75)$, see Figs. 4(a). Figure 4(b) shows a snapshot of the sound pressure level at 0.08 s. The boundary impedances are also shown in this figure, as well as the corresponding absorption coefficient values for normal incident sound waves $\alpha_n$. The instantaneous sound pressure level at the receiver points (corresponding to the energy-time curve), is plotted in Fig. 4(c). Finally, a comparison with BEM calculations is shown in Fig. 4(d). In the current case, the 2 ppλ resolution corresponds to $f = 850$ Hz. For reference, results for a calculation with an equal boundary impedance for all boundaries of $Z = 78$ ($\alpha_n = 0.05$) is also shown in Fig. 4(d), displaying the large difference in levels as a result of the difference in amount of absorption. We need to stress that the PSTD methodology has the advantage over BEM in the current configuration when inhomogeneous media effects, as temperature gradients, are to be modeled.
CONCLUSIONS

To compute sound propagation in the presence of non-staircase type boundaries with the Fourier PSTD method, an efficient time-domain wave-based prediction method, a curvilinear approach has been taken for 2D configurations. In free field, results show that a higher demand on time step and discretization is needed as the grid gets more deformed. Configurations with boundaries are also investigated. A conformal map is applied for these cases, yielding the advantage of applicability of computing pressure derivatives normal to the boundaries as in the Cartesian PSTD method. The results show that for small grid deformations, high accuracy is obtained. A reduction of accuracy however appears for moderate angular deformation, i.e. above an angular deformation of 15°. This error has two reasons. First, computation of the derivative of velocity components are approximative, i.e. the applied PSTD methodology is developed to compute the derivative of normal velocity components to the boundary. Second, for the exterior problem, the total domain consists of subdomains, causing non-smooth transitions of the derivatives the appear in the conformal map. The curvilinear PSTD methodology has finally been applied to a concert hall shaped configuration, where good agreement has been found with BEM. From the investigated configurations, it may be concluded that the current curvilinear PSTD approach holds for problems with low grid deformations. Further work includes improvement of computing velocity derivatives near boundaries and application to curved boundaries. The application of a non-conformal curvilinear mesh would be appropriate to promote improving flexibility of coupling subdomains.

REFERENCES


