3aEA6. A numerical study of non-collinear mixing of three-dimensional nonlinear waves in an elastic half-space

Zhenghao Sun, Fucai Li and Hongguang Li*

*Corresponding author's address: School of Mechanical Engineering, State Key Laboratory of Mechanical System & Vibration, Shanghai Jiao Tong University, Room 816, Building A, School of Mech. Engg., Shanghai, 200240, Shanghai, China, hgli@sjtu.edu.cn

Interaction of two non-collinear nonlinear ultrasonic waves in an elastic half-space with quadratic nonlinearity is investigated in this paper. The numerical problem is solved by a three-dimensional finite element method as subsequent work of a previous simulation using a two-dimensional semi-discrete central scheme. The nonlinear response of the resonant wave is analyzed both in time and frequency domains, and the method of non-collinear wave mixing is proved to be a promising method which is both effective and sensitive in material characterization and structure damage detection.

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INTRODUCTION

In recent years, nonlinear ultrasonic nondestructive evaluation have aroused much attention due to its sensitivity to microstructure changes, which brings about advantageous performance on detecting early fatigue damage and plastic deformation. Several types of nonlinear effects are exploited to be applicable in material characterization, especially for materials with micro-damage. These effects include harmonic generation, sub-harmonic generation, shift of resonance frequency, mixed frequency response and so on.

Interaction between intersecting waves caused by material nonlinearities was firstly investigated in 1960s, and has also been explored in experiments recently. The non-collinear wave mixing technique utilizing mixed frequency response from non-collinear directions has many prominent advantages over other nonlinear ultrasonic methods. Firstly, it provides effective selectivity of frequency, mode, direction and space. Secondly, system nonlinearity can be both independently measured and enormously eliminated. Thirdly, both single and double edge access are suitable for a valid application. However, compared with higher harmonic generation technique, the non-collinear wave mixing technique is more complex, for special conditions associated with synchronism and polarization have to be satisfied in order to generate a resonant wave.

Although analytical and experimental work on the subject of non-collinear wave mixing have been variously conducted by previous researchers, there still remains lots of blank areas pending for further investigation. A significant one is on which level at least the driving conditions, which include primary wave frequencies and amplitudes as well as the volume of the intersecting zone, should be given to generate a resonant wave that can be observed. Another focus is when and where the resonant wave comes up under appropriate circumstance. Questions like these mentioned above, which can be summarized generally as how the two incident waves interact and depend on the local microstructure. Fortunately, on this issue a numerical calculation is able to look inside and better help to understand the wave mixing phenomenon.

After our two-dimensional numerical study using a semi-discrete central scheme to simulate the propagation of two pulses in a solid medium with quadratic nonlinearity, a related implementation of expansion to three-dimension is expected so as to display a more real atmosphere and achieve a deeper inspection of the physical process. Here a numerical study with Abaqus software is implemented and the nonlinear response corresponding with a resonant wave is discussed.

MODEL FOR WAVE PROPAGATION IN AN ELASTIC HALF-SPACE WITH QUADRATIC NONLINEARITY

The three-dimensional motion is governed by a second-order hyperbolic system of partial differential equations as

\[ \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} = \rho \ddot{u}_1 \]

\[ \frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} = \rho \ddot{u}_2 \]

\[ \frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} = \rho \ddot{u}_3 \]

where \( \sigma_{rs} = \sigma_{rs}(x,y,z,t)(r,s = 1,2,3) \) represents the stress component, \( u_1, u_2 \) and \( u_3 \) represent displacement components in the x, y and z directions, respectively, and \( \rho \) represents the density of the material.

The boundary condition for the problem is given as

\[ F_y(x,y_1,z,t) = F_y(x,y_2,z,t) = -QG(t)\delta(x) \]

\[ F_z(x,y_1,z,t) = F_z(x,y_2,z,t) = 0 \]

where \( F \) is the load applied on one surface of the structure, \( \delta(x) \) is the delta function representing the load at \( z=0 \) and the input signal function is...
\[ G(t) = \begin{cases} \frac{1}{2} \sin \left( 2\pi f_j t \right) \left( 1 - \cos \left( \frac{2\pi t}{t_f} \right) \right) & \text{if } t \leq t_f \\ 0 & \text{otherwise} \end{cases} \]  

(6)

\( Q \) is the amplitude of \( G(t) \).

Here only elastic nonlinearity related with small strain deformations is considered, and the constitutive equations under this circumstance can be characterized as

\[ \sigma_{11} = (\lambda + 2\mu) u_{1,1} + \lambda (u_{2,2} + u_{3,3}) + (l + 2m) u_{1,1}^2 + l (u_{2,2} + u_{3,3}) (u_{1,1} + u_{2,2}) \]

+ \( \frac{m}{2} \left( u_{qqr} - u_{2,2} u_{3,3} \right) + n \left( u_{2,2} u_{3,3} - \frac{1}{4} \left( u_{2,2} + u_{3,3} \right)^2 \right) \)  

(7)

\[ \sigma_{22} = (\lambda + 2\mu) u_{2,2} + \lambda (u_{3,3} + u_{1,1}) + (l + 2m) u_{2,2}^2 + l (u_{3,3} + u_{1,1}) (u_{2,2} + u_{3,3}) \]

+ \( \frac{m}{2} \left( u_{qqr} - u_{3,3} u_{1,1} \right) + n \left( u_{3,3} u_{1,1} - \frac{1}{4} \left( u_{3,3} + u_{1,1} \right)^2 \right) \)  

(8)

\[ \sigma_{33} = (\lambda + 2\mu) u_{3,3} + \lambda (u_{1,2} + u_{2,1}) + (l + 2m) u_{3,3}^2 + l (u_{1,2} + u_{2,1}) (u_{3,3} + u_{2,2}) \]

+ \( \frac{m}{2} \left( u_{qqr} - u_{1,1} u_{2,2} \right) + n \left( u_{1,1} u_{2,2} - \frac{1}{4} \left( u_{1,1} + u_{2,2} \right)^2 \right) \)  

(9)

\[ \sigma_{12} = (u_{1,2} + u_{2,1}) (\mu + mu_{tr}) + \frac{n}{4} \left( (u_{1,3} + u_{3,1}) (u_{2,3} + u_{3,2}) - 2 (u_{1,2} + u_{2,3}) u_{3,3} \right) \]

(10)

\[ \sigma_{13} = (u_{1,3} + u_{3,1}) (\mu + mu_{tr}) + \frac{n}{4} \left( (u_{1,2} + u_{2,1}) (u_{2,3} + u_{3,2}) - 2 (u_{1,3} + u_{3,1}) u_{2,2} \right) \]

(11)

\[ \sigma_{23} = (u_{2,3} + u_{3,2}) (\mu + mu_{tr}) + \frac{n}{4} \left( (u_{1,2} + u_{2,1}) (u_{3,1} + u_{1,3}) - 2 (u_{2,3} + u_{3,2}) u_{1,1} \right) \]

(12)

\[ u_{tr} = u_{1,1} + u_{2,2} + u_{3,3} \]

(13)

\[ u_{qqr} = (u_{1,2} + u_{2,1})^2 + (u_{1,3} + u_{3,1})^2 + (u_{2,3} + u_{3,2})^2 \]

(14)

where \( \lambda \) and \( \mu \) are second-order elastic constants, \( l, m \) and \( n \) are third-order elastic constants.

**CONSIDERATIONS OF THE NUMERICAL SOLUTION**

The numerical problem is solved making use of Abaqus software. As the constitutive relation with third order elastic constant (TOEC) is not involved in the existing modules, work on building a novel one is necessary. User Material (Umat) is able to undertake this task. Quadratic nonlinearity of the material characterized by TOEC is expressed in a subroutine file. The geometry model is designed as 6*6*3mm³, and two separate loads are applied in the 6*6mm² plane symmetrically (Fig. 1).
FIGURE 1. Geometry model and the location of the input in a rectangular coordinate system

Modeling of an oblique incidence on the boundary can be realized in two ways. The first one is to define the specific incident direction using the analytical expression field which can be created in Abaqus. The other one makes use of phase array technique\textsuperscript{10} which is able to decide the angle of incidence by setting up the time difference between neighboring elements of the array. Both methods are applied respectively and the results presented in the following come from an input using the second one.

In the previous two-dimensional numerical research, a nearly transparent boundary is set to absorb energy and give no reflections backwards. Considering in practice there always exist reflections, and also the response of the generated resonant wave is hard to be influenced by reflected waves due to its rather strict prerequisite. It is thus found that compared with results of 2-D simulation, although those of 3-D simulation display a more complex output in the time domain owing to the boundary reflection, other frequency components do not appear apparently in the frequency domain.

SIMULATIONS AND DISCUSSIONS

Considering three polarizations as well as the formation of both summary and difference frequencies of interacting waves, there are totally fifty-four types of interacting possibilities in an isotropic solid, but most of them are forbidden to take place because of their failure to satisfy the synchronism or polarization conditions, or only the case of collinear mixing for them is allowed, thus leaving eight cases possible for a intersection induced scattering. For each case of them, the angle between two directions of primary waves at resonance can be represented by the frequency ratio and the velocity ratio, so the angle can be determined by these two positive real numbers.

In order to obtain effectively the response related with the resonant wave, the driving frequencies have to be selected appropriately. As the resonant frequency is seen as the main investigated object, it should be independent of other potentially confusable frequencies, which contain those of primary waves and higher order harmonic waves generated by the two primary waves respectively as well. Besides, since the cosine of the angle at resonance mentioned above falls into a limited interval, there is also a range of value for the frequency ratio, which has to be calculated based on the inequality. Additionally, a pair of moderate values of driving frequencies ought to be chosen with the aim of both providing sufficient energy and coordinated with the numerical iteration.

### TABLE 1. Material and Input Parameters

<table>
<thead>
<tr>
<th>$\rho$ (kg/m$^3$)</th>
<th>$\lambda$ (N/m$^2$)</th>
<th>$\mu$ (N/m$^2$)</th>
<th>$l$ (N/m$^2$)</th>
<th>$m$ (N/m$^2$)</th>
<th>$f_1$ (MHz)</th>
<th>$f_2$ (MHz)</th>
<th>$t_r$ ($\mu$s)</th>
<th>$Q$ (N/m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2700</td>
<td>4.86×10$^{10}$</td>
<td>2.43×10$^{10}$</td>
<td>-38.8×10$^{10}$</td>
<td>-35.8×10$^{10}$</td>
<td>4</td>
<td>11</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

Here an interaction of two longitudinal waves generating a transverse resonant wave is simulated. Material and input parameters are shown in Table 1. According to the principles of choosing driving frequencies mentioned above, a pair of 4MHz and 11MHz input signal with a Hamming window is selected (Fig. 2). With a fixed velocity ratio 0.5, the angle at resonance should be 132 degree, and then the direction of the resonant can be determined consequently through solving the vector triangle equation, which is roughly 78 degree with the normal direction of the inner surface. For the primary waves are travelling in the y-z plane, the resonant wave will also propagate within the plane.
The observation points can be chosen based on the generation location and scattering direction of the resonant wave. Characteristic of the resonant wave is accessible through extracting the features of these points.

**FIGURE 2.** Input signals of two beam arrays with a frequency component of (a) 4MHz and (b) 11 MHz, respectively.

The exciting sources composed by two beam arrays are placed on one surface of the structure, and two primary waves meet with each other at the perpendicular bisector of two sources. The equivalent length of the interaction zone, which depends on the length of the beam arrays, should be both large enough to realize the generation of the resonant wave and small enough compared to the distance from the zone to the observation points. Nodes in the zone where the resonant wave arises and propagates are investigated and several typical ones of them are presented as follows. Displacements and velocities of one node varied with time are directly obtained from the History Output, and the related frequency domain response can be drawn by FFT transformation. Since two driving frequencies are not the same, it appears a complicated shape once the two primary waves meet each other for the first time (Fig.3a), and it’s hard to distinguish one pulse response from the other. When the resonant wave propagates a little further away from the interaction zone, which is nearer to the source of 4MHz than the 11MHz one, the impact of the pulse of 11 MHz becomes more unremarkable, and it displays an outline of the 4MHz one at the front part (Fig.3b&c). As the resonant wave travels further away, effects of both primary waves get weakened, and the previous 4MHz wave pack gradually vanishes, leaving an intricate pattern again (Fig.3d).
FIGURE 3. Numerical solution of the particle velocities \( \dot{u}_r \) in a nonlinear half-space at different locations along the direction of the resonant wave propagation: (a) \( s = 0 \)mm, (b) \( s = 0.6 \)mm, (c) \( s = 2.4 \)mm, (d) \( s = 2.7 \)mm.

The frequency domain plays a more significant role in analyzing the nonlinear effect of mixing waves, which provides a different perspective from traditional linear ultrasonics. The period of the whole flight time contains two consecutive parts, the first of which starts from the source of waves to the interaction zone, and the second from the zone to the observation point. For propose of looking into the scattered wave more clearly, a certain section ought to be chosen appropriately from the time domain signal before processing FFT. Generally, the period to be processed can be identical with the second part mentioned above, or in an operation extends a bit longer than that at both ends.

At the very beginning of the propagation of the resonant wave, the two distinct components appearing in the frequency domain just come from primary waves, which are 4MHz and 11MHz respectively, and no other components can be obtained obviously (Fig.4a). Along with the travelling of the resonant wave, the difference frequency component that is 7MHz slowly emerges, and also the amplitudes of two driving frequencies make change due to the wave front moving of intersecting waves (Fig.4b). As the scattered wave further away from its birthplace, its frequency component continues to exist and the ratio between itself and primary ones increases with propagating distance (Fig.4c\&d).
FIGURE 4. FFT of the particle velocities \( \dot{u}_p \) in a nonlinear half-space at different locations along the direction of the resonant wave propagation: (a) \( s = 0 \) mm, (b) \( s = 0.6 \) mm, (c) \( s = 2.4 \) mm, (d) \( s = 2.7 \) mm.

CONCLUSIONS

Based on preceding study on two-dimensional numerical calculation of non-collinear wave mixing, a three-dimensional case is investigated using a secondary development of Abaqus solver with Umat. A phase array technique is applied here to stimulate two non-collinear primary waves which enables the generation of a resonant wave under strict conditions. Observation nodes are cautiously chosen in accordance with the propagating path of the third wave. The analysis of the results proves the existence of the resonant wave and manifests that the frequency amplitude changes with locations. The numerical research stands as a visualized approach to look into the nonlinear effect caused by a non-collinear wave mixing, thus will provide guidance to the related experiments. Further exploration of this issue is still under way.

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