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3pEA3. On the benefits of Debye series for modeling the ultrasonic propagation in a waveguide
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This short communication aims to compare the Debye series with a modal series for the case of the propagation of a bounded beam in an embedded waveguide. The calculation can be achieved through several methods. These methods are described briefly and the most preponderant references are given. The Modal Expansion Method which is among these method is compared to the Debye series which is combined with the Integral Transform Method. The benefits of the Debye series for the case of embedded waveguides are highlighted.

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INTRODUCTION

Theoretical modeling is needed in Non Destructive Testing and Evaluation (NDT&E) via guided waves. It provides a fast prediction of wave propagation behavior in a waveguide. Different physical parameters which may influence the propagation can be tested for optimization, and corresponding experiments are consequently more efficient. Therefore, these preliminary theoretical tests reduce the cost of such an experiment.

In a waveguide, the principal problem is to determine the field generated by an emitter at a desired point located in the plane of the receiver. The propagating wave interacts with the walls of the waveguide which make it dispersive and multimodal. Figure 1 illustrates this idea.

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INTEGRAL TRANSFORM METHOD (ITM)

The application of an appropriate integral transform changes the problem from a set of partial differential equations in space-time to a linear system of equations in frequency-wavenumber. Its use dates back to the 1950’s with J. Adem [1] and Folk et al. [2]. The former applied it to two bars: one being infinite and the other, semi-infinite. Both were subject to an axial loading of type: \( \delta(z) \exp(\lambda z) \). The latter was used for the study of a semi-infinite bar with a mixed condition limits and no radial variation. Having found problems in the reverse integration, they resorted to asymptotic expansions for the successful application of the residue theorem. Ten years later, in 1967, Sherman [3] gave a new form to this method and applied it to a radial variable dependent load. The method is also called "angular spectrum method" or "waves decomposition method" for planar or cylindrical coordinates (vis-à-vis of the studied geometry). It has been extended to the field of acoustics and several authors have used it in a medium described by a Cartesian coordinate system [4-7]. This method, based on the Fourier transform in Cartesian coordinates or the Hankel transform in the cylindrical coordinate system, can evaluate the distribution of the sound field in any plane when its distribution is known in a plane which is parallel to it. The first plane may be that of the source; the second plane may correspond to the receiver. The field acquired by the latter is the combination of that emitted by the source and a transfer function describing the propagation. It will be shown that this function can be expressed in terms of the number of interactions (or reflections) with the wall of the guide.

MODAL EXPANSION METHOD

The principle of this method is based on writing the function describing the source of a product of two functions. One is the modal behavior of the acoustic wave and the other models the propagation of the disturbance that is a function of time and space. It can be expressed as follows [8, 9]:

FIGURE 1. Illustration of the problem statement.
where the symbol "prime" indicates that the term \( m = 0 \) is omitted. This decomposition assumes that the guided modes are mutually orthogonal and demonstrates that the energy of a mode cannot be transferred to another mode. This orthogonality condition is necessary in this method. Determining the amplitude of each mode \( A_{mn} \) is performed from the boundary conditions applied to the end of the guide.

This method is not desirable in the case of a coated guide when the shear velocity in the waveguide is greater than that of the coating, as described in [10]. For a specific configuration (steel bar embedded in a grout), Simmons [10] showed that the guided modes were attenuated. More specifically, all modes of the structure have energies receding from the bar to the coating, and therefore the waveguide is leaky.

This method is impractical since no orthogonality relation linking modes is currently known. Therefore, the method based on the modal decomposition is difficult to apply to the problem of wave propagation in a waveguide embedded in an infinite matrix.

RAY METHOD

This method can be used in the frequency domain as well as in the time domain. In the time domain, there is the theory of virtual sources (sometimes called fictitious image theory [11]). Mathematically, this approach is relatively simple and accurate. Many studies [12, 13] have used this method. Its disadvantage is that it is only applicable to point sources. In the frequency domain, there is the Debye series. It will be shown that by combining this function with the ITM, guides excited by a surface source can be studied.

TRANSPARENT BOUNDARY CONDITION APPROACH

This resolution requires heavy mathematical formalism but can take advantage of the discontinuity condition applied to the interface. This formalism requires also many integral transforms to solve various differential equations and the major difficulties of solving the problem are then to perform the inverse transformations [11-17].

COMBINED METHODS

We have quoted various methods dealing with waveguides. Each has its own advantages and disadvantages. With one or the other, it is incapable of solving certain problems for various reasons. Whether one or the other is used, some problems remain insoluble for different reasons. The combination of certain methods may be a remedy to these deficiencies.

Modal expansion / Ray-mode

The combination of the modal method and the ray method was used longtime ago [18-21] in order to describe the wave field generated by a point source. To transform modes into rays, the concrete form of the eigenvectors (modes) can be used, which makes possible the replacement series of normal modes by an integral ("contour Watson's integral " for example). Then its asymptotic behavior is examined by applying the residue theorem and the stationary phase. To transform the mode into rays in a homogeneous ocean of constant depth [18] and in a flat waveguide [20], the Poisson summation formula is used. With the idea of the construction of asymptotic values of the normal modes by WKB method, this technique can be successfully extended to the case of vertically inhomogeneous waveguides [18-20]. This technique has been extended by Smyshlyaev [21] to the case of a 3D waveguide with variable section.

ITM/ Ray-mode

The Debye series was used to express the transfer functions (or general reflection coefficient) based on an infinite number of local reflection coefficients. The combination of this series with the ITM has been used by Deschamps and Chengwei [22] for the study of propagation in an isotropic or anisotropic plate [23]. Danthez [24]
adapted this method to study the propagation of a wave beam bounded in time and space in a cylindrical bare waveguide. He showed that the velocity field prevailing in the guide can be expressed as the multiplication (in the frequency domain) of elementary solutions of the equations of propagation with transfer functions describing the multiple reflections of the wave inside the waveguide. Two approaches were used to calculate the transfer function. The analysis by local interaction wave / waveguide wall described this solution as a series of partial waves through the local reflection coefficients and allowed immediate physical interpretation of the result.

From the theoretical analysis by Danthez, Grimault [25] developed an analytical model including a laterally infinite medium coating the cylindrical guide

\[ u(r, z, t) = \int_{-\infty}^{+\infty} \int_{0}^{+\infty} \tilde{U}(k_z, \omega) F(k_z, \omega) P(r, k_z, z, \omega, t) \, dk_z \, d\omega \]  

(2)

where \( \tilde{U} \) is the spectrum of the source, \( P \) is a term describing the propagation according to the radial and axial directions and time and \( F \) is the transfer function. This function is closely related to the waveguide: It depends on its geometry, its material and its state (coated or uncoated). It may also depend on the number of ray interaction with the wall of the waveguide. To do this, the transfer function was expressed using the Debye series:

\[ F(k_z, \omega) = \sum_{m=1}^{+\infty} R_{m-1} \]

(3)

where \( R \) is the coefficient of local reflection and \( m \) denotes the \( m^{th} \) reflection. In [25], this function takes into account the transmission in the surrounding medium. The developed 2D formalism was generalized [26, 27] to the case of 3 dimensions. In this case, equation (2) becomes:

\[ u(r, \theta, z, t) = \sum_{n} \int_{-\infty}^{+\infty} \left[ \int_{0}^{+\infty} \tilde{U}(k_z, \omega) F_n(k_z, \omega) P_n(r, \theta, k_z, z, \omega, t) \, dk_z \right] \, d\omega \]

(4)

Moreover, and from a numerical point of view, the integral over \( k_z \), which is due to the Hankel transform, cannot be carried out quickly. It consumes a lot of memory and need long CPU time, which can be amplified by applying the sum (Eq. 4). If for physical reasons, the expression of this field is desired in the \((k_z, \omega)\) domain, one additional integral (over \( z \)) is also needed.

**COMPARAION OF ITM & MEM**

In order to compare the two methods (ITM and MEM), we can write, in a general form, according to the two equations (1) and (4):

- in the time domain

\[ u(r, t) = \sum_{n=0}^{+\infty} u_n(r, t) = \int_{-\infty}^{+\infty} \left[ \sum_{m=0}^{+\infty} M_m(r, \omega) \right] e^{-i\omega t} \, d\omega \]

(5)

- and in the frequency domain

\[ U(r, \omega) = \int_{-\infty}^{+\infty} \left[ \sum_{n=0}^{+\infty} u_n(r, t) \right] e^{i\omega t} \, dt = \sum_{m=0}^{+\infty} M_m(r, \omega) \]

(6)

where \( n \) denotes the \( n^{th} \) reflection with the boundary of the waveguide in the Debye series and \( m \) is the \( m^{th} \) mode in the modal series.
As has been stated, through the Debye series, we describe the waveform (time signal) from an infinite number of reflections on the wall of the waveguide (interactions with the interface waveguide / coating, if the waveguide is coated). According to Eq. (5) (Debye series), at each iteration, information in the time signal is added. Physically, each reflection gives rise to a wave train (or pulse, primary or secondary) with an appropriate delay with respect to causality. This wave train is zero for any time less than $t_{0n}$: $u_n = 0$ $t < t_{0n}$, its arrival time. Therefore, by increasing the order $n$, the precision of the time signal increases over a duration which is increasingly larger since $t_{0n}$ increases with $n$. If we sum up the $n$th reflection, it is certain that the calculation is exact for any time prior to the arrival of this final reflection: $t < t_{0n}$ (see fig. 2, left).

In addition, each input series affects the total width of the spectrum. For an exact calculation of the spectrum, it is necessary to sum all the possible reflections. In other words, by increasing $n$, the spectrum becomes more accurate. This accuracy is governed by the number of terms of the Debye series. Thus, we must sum to infinity for an exact spectrum including both high frequency and low frequency content.

In contrast, the spectrum is more precise at low frequency when using modal series. Indeed, for all frequencies $f < f_{cm}$ ($f_{cm}$ being the cutoff frequency of the $m$th mode), the higher order modes do not affect this spectral band. Therefore a duality exists between the Debye series and modal series. In particular, the cutoff frequencies $f_{cm}$ are for the modal series as the arrivals time for the Debye series. So, if we sum up the $m$th mode, it is certain that the calculation is exact for any frequency prior to the cutoff frequency of the mode: $f < f_{cm}$ (see Fig. 2, right).

To elucidate, the calculation of the Green's function can be a good example. This function is the time response to an excitation by a Dirac distribution. Its spectrum is therefore a constant for $f \in [-\infty, +\infty]$. Modal theory is not suitable because it is essential to achieve an infinite sum to have the temporal precision which is necessary to represent the distributions. However, with the Debye series, the first contributions may be sufficient to have an exact match within the desired time domain, since reflections / refractions multiple of the discontinuous fields are calculated separately and with the required frequency accuracy.

**CONCLUSION**

The MEM is particularly suited to the frequency domain. The ray method, in turn, is adapted to the time domain. Both methods can be complementary in the case of uncoated guides. But as the MEM is not easily applicable in the case of a coated guide, ITM combined with ray theory is advised to resolve this kind of problem.

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