4pEAb5. Acoustical impedance characterization of liners using a Bayesian approach

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A characterization method of acoustic treatment composed by a perforated plate and a honey comb layer is described. Those treatments are generally used as liners for aircraft nacelle and are characterized by their impedance. Empirical models or Kundt’s tube experiments are generally used in order to characterize impedance under specific conditions which are not representative of physics in aircraft liners (plane wave instead complex waves, normal incidence instead grazing incidence...). An inverse method using Bayesian approach was developed to define specific parameters describing impedance of treatments under similar functional conditions. For this, an analytical solution is implemented in order to evaluate the posterior probability density function of parameters and an evolutionary Markov Chain Monte Carlo method (eMCMC) is implemented to explore probability density space. The validation of the inverse method is realized on a simulated data and a real case based on experimental data is presented.

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INTRODUCTION

Locally reactive acoustic liners such as honeycomb structures with perforated panels can be modeled with a surface impedance in standard numerical models. However the characterization of this impedance is not always straightforward. Empirical models or standing wave tube measurements are generally used to get the behavior of these acoustic treatments. Unfortunately, these methods provide only an evaluation of the impedance under specific conditions. A large dispersion is indeed observed between the empirical models and the measurements as shown in Figure 1.

![Figure 1: Absorption coefficient comparison between measured data and two empirical models - Liner characteristics are recapped in Table 1.](image)

Moreover, the conditions of use can change significantly the acoustic liners behavior. Santana et al. [1] showed for example the influence of the transition between a rigid and a treated wall and the effect of a flow on the sound propagation that was observed experimentally and not in the simulations. Schultz et al. [2] highlighted also the effect of oblique incident waves, i.e. higher order modes, on the surface impedance and showed some discrepancies with the measurements obtained with the Two-Microphone-Method (TMM). Finally the classical empirical models fail to predict accurately the surface impedance of acoustic liners under real conditions (see also Ref. [3]).

A characterization of locally reactive acoustic liners is presented here. Starting from a surface impedance measurement of a honeycomb structure with a perforated panel, an inverse method based on Bayesian approach is used to return the surface impedance taking in consideration the real conditions of use. A rectangular duct treated by a liner on its upper face is considered here (see Figure 3). These conditions are similar to the experiment presented in Figure 2. In the first step, the inverse method of characterization is presented on simulated data with an added random noise. This inverse method requires a direct model to predict the pressure at some microphone positions with any surface impedance. The model used in the following is based on the Mode-Matching method presented by Nennig et al. [4]. The method is indeed very efficient and sufficiently fast for multi-modal propagation.

From the direct analytical model, the Bayes’ rule is then used to get the posterior probability density function of the estimated impedance. Chazot et al. [5] showed that the likelihood function could be evaluated easily by using the central limit theorem. In this process the impedance is parametrized with a rational function and a prior information is added on each parameter to regularize the inverse method. Finally a Cost Function is obtained and explored with an eMCMC algorithm. This method provides not only the best set of parameters but also some statistical information for each parameter.
ANALYTICAL MODEL OF SOUND PROPAGATION IN A DUCT

In order to use the Bayesian inference to estimate the surface impedance, it is necessary to predict the pressure at each microphone position for any frequency-dependent impedance $Z(\omega)$.

An analytical model is therefore implemented and enables to simulate the propagation of complex acoustical waves in a rigid (I)/treated (II)/rigid (III) rectangular duct with an anechoic termination on each side (see Figure 3).

The model is based on a modal expansion and a mode-matching method. Acoustic modes are thus calculated in the rigid and the treated part of the duct.

Acoustic Modes Calculations

The time-harmonic acoustic wave propagation (assuming a time dependency in $e^{-i\omega t}$) is described by the Helmholtz equation:

$$\frac{\partial^2 P}{\partial X^2} + \frac{\partial^2 P}{\partial Y^2} + \frac{\partial^2 P}{\partial Z^2} - k^2 P = 0,$$

where $P$ denotes the acoustic pressure and $k = \omega/c$ is the wave number.

The boundary conditions are given by:

$$\frac{\partial P}{\partial n} = 0 \quad \text{on the rigid wall, and}$$

$$\frac{\partial P}{\partial V} = Z \quad \text{on the treated wall.}$$

Using the separation of variables, the acoustic pressure for the mode $(mn)$ in the rectangular duct writes:

$$P_{mn}(x, y, z) = A_{mn} \cos(k_m X) \cos(k_n Y) e^{ik_m z}$$

For the rigid duct, the wave numbers are given by:

$$k_m = \frac{m\pi}{a},$$

$$k_n = \frac{n\pi}{b},$$

$$k_{mn} = \sqrt{k^2 - k_m^2 - k_n^2}.$$

For the treated duct, the acoustical pressures verify the Ingard-Myers condition at $y = 0.1m$:

$$\frac{\partial P}{\partial n} = -\frac{i}{kZ} \left( M \frac{\partial}{\partial z} - ik \right)^2,$$
with $Z$ the dimensionless impedance. Without flow, ie. $M = 0$, this condition leads to:

$$k_n \sin(k_n b) - \frac{ik}{Z} \cos(k_n b) = 0$$  \hspace{1cm} (6)

By replacing $k_n$ by $\sqrt{k^2 - k_m^2 - k_{mn}^2}$, this equation becomes:

$$\sqrt{k^2 - k_m^2 - k_{mn}^2} \sin(\sqrt{k^2 - k_m^2 - k_{mn}^2} b) + \frac{ik}{Z} \cos(\sqrt{k^2 - k_m^2 - k_{mn}^2} b) = 0 .$$  \hspace{1cm} (7)

For each order $m$, the wave number $k_m$ is known and the wave number $k_{mn}$ is calculated by using the Newton-Rhapson method. The rigid mode solution is then considered as initial guess in the Newton-Rhapson method and increased gradually to get the real impedance (more details are given in Ref. [6]). Finally the wave number $k_n$ is also obtained thanks to the dispersion equation (Eq. 4c).

**Mode Matching**

The pressures in the rigid and treated parts are coupled with a mode-matching technique (see Ref. [4]). This method is presented briefly here.

**Matching Conditions**

The acoustical pressure in each duct section $s=I,II,III$ is expanded over its modal basis:

$$p^s = \sum_{mn}^K A^s_{mn} \phi^s_{mn}(x, y) e^{ik_{mn}z} + A^s_{mn} \phi^s_{mn}(x, y) e^{-ik_{mn}z} ,$$  \hspace{1cm} (8)

with $A^s_{mn}$ the modal amplitudes of the incident and reflective waves in the section $s$, and $\phi^s(x, y)_{mn}$ the related $mn$th pressure mode shape. The acoustical velocity writes in the same way:

$$v^s = \sum_{mn}^K A^s_{mn} \varphi^s(x, y)_{mn} e^{ik_{mn}z} + A^s_{mn} \varphi^s(x, y)_{mn} e^{-ik_{mn}z} ,$$  \hspace{1cm} (9)

with $\varphi^s(x, y)_{mn}$ the $mn$th velocity mode shape in the duct section $s$. These mode shapes and modal amplitudes are finally concatenated in a matrix form such as $\phi^s_{p} = [\phi^s_{p,1} \ldots \phi^s_{p,K}]$, $\phi^s_{v} = [\phi^s_{v,1} \ldots \phi^s_{v,K}]$, and $A^s_{p} = [A^s_{1} \ldots A^s_{K}]$.

At the plane $z = z_1$, the pressure on the left side is given by:

$$p^I = \Phi^I_{p} A^I + \Phi^I_{p} A^I ,$$  \hspace{1cm} (10)

while the pressure on the right side at the same position writes:

$$p^{II} = \Phi^{II}_{p} A^{II} + \Phi^{II}_{p} A^{II} .$$  \hspace{1cm} (11)

The pressure continuity at the interface, ie. $p^I = p^{II}$, gives then a new system of equations between the unknown modal amplitudes in each section. Similar expressions can be written at each interface and also for the acoustical velocity. Finally, the pressure and velocity continuities at each interface between the rigid and treated sections of the duct lead to the following matching equations:

$$p^{I}(x, y, z_1) = p^{II}(x, y, z_1) ,$$  \hspace{1cm} (12a)
\[ v^I(x, y, z_1) = v^{II}(x, y, z_1), \quad (12b) \]
\[ p^{II}(x, y, z_2) = p^{III}(x, y, z_2), \quad (12c) \]
\[ v^{II}(x, y, z_2) = v^{III}(x, y, z_2). \quad (12d) \]

This matching procedure enables at the end to solve the global problem of sound propagation in the duct.

**Numerical Implementation**

In practice, the matching conditions are formulated in a variational form and integrated in the cross sectional area of the transition planes. For example, the conditions for the plane \( z = z_1 \) write:

\[
\int \int \Phi_p p' dx dy = \int \int \Phi_p^{II} dx dy, \quad (13)
\]
\[
\int \int \Phi_v v' dx dy = \int \int \Phi_v^{II} dx dy. \quad (14)
\]

According to Nennig et al. [4] the most efficient choice for the test functions \( \Psi \) are the rigid wall mode shapes. The unknowns of the problems are the modal amplitudes \( A^{I\pm}, A^{II\pm}, A^{III\pm} \) while the modal amplitudes \( A^{I\pm} \) and \( A^{III\pm} \) are given as inputs. A set of propagating modes are indeed applied in the first section, and an anechoic termination is considered at the third section end. The following scattering system is finally obtained:

\[
D_1 \begin{pmatrix} A^{I-} \\ A^{I+} \end{pmatrix} = D_2 \begin{pmatrix} A^{I+} \\ A^{I-} \end{pmatrix}, \quad (15a)
\]
\[
D_3 \begin{pmatrix} A^{III+} \\ A^{III-} \end{pmatrix} = D_4 \begin{pmatrix} A^{III-} \\ A^{III+} \end{pmatrix}. \quad (15b)
\]

An iterative method is used to solve this system. The amplitudes \( A^{III-} \) are set to zero in the first iteration. The amplitudes \( A^{III+} \) are then calculated with the Eq. 15a and replaced in Eq. 15b to get the new estimated amplitudes \( A^{III-} \). This iterative process is realized until a convergence criterion is reached.

**Validation**

The mode-matching model is now compared with a Finite Element Model. Quadratic T10 elements are used and anechoic conditions are imposed with DtN’s conditions (see Ref. [6]). The measured surface impedance presented in Figure 6 is directly imposed on the treated wall. The comparison between the analytical model and the FEM is illustrated in Figures 4 & 5 for multi-modal propagation at 2000Hz in the rectangular duct treated at \( y = 0.1m \) between positions \( z_1 = 0m \) and \( z_2 = 0.5m \). The analytical model is therefore validated and is now used as reference in our characterization method.
Bayesian Identification Method

In the identification process described here, the impedance must be parametrized with an analytical model. As explained in the beginning, the classical empirical models fail to represent the real behavior of a liner composed of perforated and honeycomb panels. We choose therefore a rational function with 7 parameters $\theta = (a_0, a_1, b_0, b_1, c_0, c_1, d_0)$ to parametrize the surface impedance such as:

$$Z(f) = \frac{a_1 \times f + a_0 + i(b_1 \times f + b_0)}{c_1 \times f + c_0 + id_0}.$$  \hspace{1cm} (16)

The main advantage of the Bayesian characterization method is to provide a probability density function (pdf) on each estimated parameter $\theta_j$. Consequently the method does not only provide the best set of parameters associated with the optimal impedance, but also the uncertainty on each parameter, their influence on the cost function, and their mutual correlations.

Cost Function

The conditional pdfs of the parameters $\theta$ given the measured pressures $\tilde{P}_{ik}$ are noted $p(\theta|\tilde{P}_{ik})$, where $i$ is the microphone index and $k$ the frequency line ($f_k$). These pdfs are also called the posterior pdfs $p(\theta|\tilde{P}_{ik})$. Using the Bayes’s Theorem, the posterior pdf is expressed as:

$$p(\theta|\tilde{P}_{ik}) = \frac{p(\tilde{P}_{ik}|\theta)p(\theta)}{p(\tilde{P}_{ik})}. \hspace{1cm} (17)$$

The pdf $p(\tilde{P}_{ik})$ is called the evidence. It does not depend on the parameters $\theta$ and is therefore constant in Eq. (17). The prior pdf $p(\theta)$ is the user’s knowledge on the parameters before the experiment. Finally the likelihood function $p(\tilde{P}_{ik}|\theta)$ is the probability to have the measurement knowing the parameters. It can be evaluated from the direct problem with the analytical model presented previously. The measured pressures are hence written as the sum of a predicted pressure with a set of parameters $\theta$ and a random noise $N_{ik}$:

$$\tilde{P}_{ik} = P_{ik}(\theta) + N_{ik}. \hspace{1cm} (18)$$

The likelihood pdf as therefore the same pdf as the random noise $N_{ik} = \tilde{P}_{ik} - P_{ik}(\theta)$. This pdf is expressed here with a complex Gaussian pdf by invoking the Central Limit Theorem (Ref. [5]). The prior pdfs of the inferred parameters are very important in the Bayesian approach since it leads to the stability and the uniqueness of the solution. The choice of the prior pdf $p(\theta)$ can be made from the user’s knowledge, or it can be assessed with a measurement at normal incidence in a standing wave tube.
TABLE 1: Acoustic Liner Characteristics

<table>
<thead>
<tr>
<th>Perforated plate</th>
<th>Thickness</th>
<th>0.5mm</th>
</tr>
</thead>
<tbody>
<tr>
<td>Perforation diameters</td>
<td>0.4mm</td>
<td></td>
</tr>
<tr>
<td>Plate porosity</td>
<td>3.5%</td>
<td></td>
</tr>
<tr>
<td>Honeycomb panel</td>
<td>Depth</td>
<td>20mm</td>
</tr>
</tbody>
</table>

At first approximation, \( p(\theta) = \prod p(\theta_j) \). All the parameters are thus independent \textit{a priori} but not necessarily \textit{a posteriori}. A Gaussian pdf is also used in a first step to characterize each \( p(\theta_j) \). With these assumptions, the posterior pdf reduces to the following expression called the \textbf{Cost function}:

\[
p(\theta | \tilde{P}_{ik}) \propto \prod_j \frac{1}{\sqrt{2\pi \sigma_{\theta_j}}} \exp\left(-\frac{|\theta_j - \tilde{\theta}_j|^2}{2\sigma_{\theta_j}^2}\right) \prod_k \frac{1}{\pi^n \prod^n_{i=1} \sigma_{ik}^2} \exp\left(-\sum_i^n \frac{|P_{ik} - \tilde{P}_{ik}|^2}{\sigma_{ik}^2}\right),
\]

with \( \tilde{\theta}_j \) the \textit{a priori} value of the parameter \( j \), \( \sigma_{\theta_j} \) its the standard deviation, \( \sigma_{ik} \) the noise standard deviation, and \( n \) the number of microphones. The optimal parameters are obtained at the maximum of the cost function or in a more convenient way, when the negative logarithm of the cost function is minimized. This leads to the new \textbf{cost function}:

\[
F(\theta) = \sum_j \frac{|\theta_j - \tilde{\theta}_j|^2}{2\sigma_{\theta_j}^2} + \sum_k \sum_i^n \frac{|P_{ik} - \tilde{P}_{ik}|^2}{\sigma_{ik}^2}
\]

\textit{Prior Information}

A treatment composed of a micro-perforated plate and a honeycomb panel is considered here. Its characteristics are presented in Table 1.

The parameters \( \theta \) related to this liner must be evaluated \textit{a priori}. The normal surface impedance is therefore measured in a standing wave tube, and the rational function 16 is adjusted on this measurement to get a set of initial parameters. Figure 6 shows the measured normal impedance and the rational function with the following adjusted initial parameters: \( a_1 = 2.5886.10^{-6} \); \( a_0 = 0.0574 \); \( b_1 = -1.4059E-5 \); \( b_0 = -0.3001 \); \( c_1 = 5.2685E-4 \); \( c_0 = -0.9897 \); \( d_0 = -0.1434 \). Finally a Chi2 distribution is defined around this set of initial parameters to get the prior pdf \( p(\theta) \).

\textbf{EMCMC Method}

An Evolutionary Markov Chain Monte Carlo (EMCMC) technique based on \textit{parallel tempering} [7] is used to explore the space of the parameters pdf. This method combines
a Genetic Algorithm, a Monte Carlo method and several Markov Chains to explore the search domain in a smart way. The idea behind EMCMC methods is to generate a range of Markov Chains with a Metropolis-Hastings algorithm that converges to the posterior pdf. At each generation in the Markov Chains, a Genetic Algorithm is used to exchange information between the different chains by crossovers and permutations. This avoids to be trapped in a local minimum of the cost function.

RESULTS OBTAINED ON SIMULATED DATA

The method is now evaluated on the problem presented in Figure 3. A multimodal propagation in the duct with unitary incident coefficient is simulated in the frequency range $[1500 - 3000]$ Hz with 30 frequency lines. The acoustic liner is known to be more efficient at these frequencies.

The pressure is calculated at 72 microphone positions: 36 before and 36 after the acoustical liner. The simulated pressures on which the Bayesian approach is done are obtained by applying a 20% random noise on the previous 72 calculated pressures. The standard deviations $\sigma_{\theta_j}$ and $\sigma_{\theta_k}$ are set to 5% and 10% respectively. These choices are made in order to balance the impact of the likelihood function with the prior pdf on the posterior pdf.

To explore the parameter space, the EMCMC is applied with 10 individual chains on 1500 generations. The initial parameters values are defined far from the values used in the simulated data in order to evaluate the efficiency of the method to recover the optimum.

The posterior pdfs are presented in Figure 7. The best set of parameters is also compared with the initial parameters. Almost all the parameters are well identified with the eMCMC method. The posterior pdfs are indeed closely distributed around the reference values. Figures 8 and 9 show also two pressures in the duct calculated with the reference parameters and with the best set of parameters. The estimated parameters give a very good estimation of the pressure in the duct. The characterization strategy is therefore validated with this simulated experiment.

**FIGURE 7:** Posterior pdf for each parameters - Red vertical line: reference value of parameters - Black vertical line: best individual value.
CONCLUSION

An inverse method to characterize the surface impedance of acoustic liners has been presented and validated with simulated data in this paper. This method is based on an analytical model of sound propagation in a duct with an acoustic liner. The Bayes' rule is also used to calculate a cost function from a likelihood pdf and a prior pdf. An eMCMC algorithm is finally used to explore the posterior pdf space. This optimization method allows to find a set of optimal parameters that characterizes the surface impedance of the liner, but also the full posterior pdfs of each parameters.

REFERENCES


