4aMU10.  A thermoviscous tube propagation model suitable for time domain analysis

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Modeling acoustic propagation in tubes including the effects of thermoviscous losses at the tube walls is important in thermoacoustics, in hearing aid modeling and in modeling wind musical instruments. Frequency dependent impedances for a tube transmission line model in terms of the so-called thermal and viscous functions are well established, and form the basis for frequency domain analysis of systems that include tubes. However, frequency domain models cannot be used for systems in which significant nonlinearities are important, as is the case with the pressure-flow relationship through the reed in a woodwind instrument. This paper describes a tube model based on a continued fraction expansion of the thermal and viscous functions. The expansion can be represented as an analog circuit model which allows its use in time domain system modeling. A simple model of a clarinet-like oscillation will be shown.

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INTRODUCTION

This paper describes acoustic propagation in a rigid tube with circular cross section as shown in Figure 1, including the effects of thermal and viscous losses at the walls of the tube. The acoustic variables in this analysis are pressure $p$ and volume velocity $u$. The subscripts shown in Figure 1 indicate the values of the variables at the two ends of the tube. There is a long history of the analysis of propagation in tubes including thermal and viscous effects that is well summarized by Stinson. Using the language of Benade, Keefe, and Swift, the circular acoustic transmission line has the series impedance $Z$ and shunt admittance $Y$ per unit length given by

$$Z = \frac{j\omega \rho}{\pi a^2 (1 - f_v)}$$

$$Y = \frac{j\omega \pi a^2}{\rho c^2} \left[ 1 + (\gamma - 1) f_t \right]$$

where the medium has density $\rho$ and sound speed $c$, $\gamma$ is the ratio of specific heats, $\omega$ is the angular frequency of the wave, and the tube has radius $a$. For any tube geometry, the thermal and viscous functions $f_v$ and $f_t$ are of the same form. For cylindrical tubes they are given by

$$f_{v,t} = \frac{2J_1(z_{v,t})}{z_{v,t} J_0(z_{v,t})}$$

where $\delta_v$ and $\delta_t$ are the viscous and thermal boundary layer thicknesses respectively, given by

$$\delta_v = \sqrt{\frac{2\nu}{\omega}}$$

$$\delta_t = \sqrt{\frac{2\kappa}{\omega \rho C_p}}$$

and $\nu$ is the kinematic viscosity of the fluid, $\kappa$ is its thermal conductivity, and $C_p$ is its specific heat at constant pressure. With these definitions, the infinitesimal tube can be represented by the impedance network shown in Figure 2.

**FIGURE 1.** The tube with rigid walls has radius $a$ and length $\ell$.

**FIGURE 2.** The impedance network representation of a tube of infinitesimal length.
For tubes whose radius is large compared to the boundary layer thicknesses, \( f_{v,t} \to 0 \) and the infinitesimal tube may be adequately modeled by the lossless network of Figure 3. Note that a tube of finite length would require a number of cascaded sections of either Figure 2 or 3 with the individual section length much less than a wavelength at the highest frequency of interest. For lossless transmission, the method first described by Branin\(^5\) for electrical transmission lines is much faster and has been implemented in many computational codes. Furthermore, when a transmission line includes frequency independent losses, a number of similar methods are appropriate\(^6\)-\(^8\). All of these methods can be used in either time domain or frequency domain analyses. However, when the losses are frequency dependent, as are the thermoviscous losses in Equations 1 and 2, a different method is necessary.

**FIGURE 3.** For \( a \gg \delta_{v,t} \), the tube is lossless and is modeled by this lumped network.

## TUBE MODEL WITH THERMOVISCOUS LOSSES

This section will develop a complete model for the tube with thermoviscous losses. The development starts with by finding models for the impedance per unit length \( Z \) and the admittance per unit length \( Y \), and then assembling the finite length of tube as the cascade of a number of short segments.

### The Impedance per Unit Length \( Z \)

Equation 1 can be rewritten as

\[
Z \, dl = \frac{j \omega \rho \, dl}{\pi a^2 (1 - f_v)} = \frac{j \omega \rho \, dl}{\pi a^2} \left( \frac{1}{1 - f_v} \right)
\]

\[
= \frac{j \omega \rho \, dl}{\pi a^2} - \frac{j \omega \rho \, dl}{\pi a^2} \frac{1}{1 - \frac{2J_1(z_v)}{z_v J_0(z_v)}}
\]

The Bessel function ratio can then be expanded as a continued fraction\(^9\)

\[
Z \, dl = \frac{j \omega \rho \, dl}{\pi a^2} - \frac{j \omega \rho \, dl}{\pi a^2} \frac{1}{1 - \frac{2}{z_v} \frac{1}{2 - \frac{1}{z_v}}} \frac{1}{1 - \frac{4}{z_v} \frac{1}{2 - \frac{1}{z_v}}} \frac{1}{1 - \frac{6}{z_v} \frac{1}{2 - \frac{1}{z_v}}}
\]

which can be simplified to
\[ Z d\ell = j\omega L_0 + R_v + \frac{1}{j\omega L_0 + \frac{1}{2R_v + \frac{1}{j\omega L_0 + \frac{1}{3R_v + \frac{1}{j\omega L_0 + \ddots}}}}} \]  

(8)

where \( L_0 = \rho d\ell / \pi a^2 \) and \( R_v = 8\pi pv d\ell / \left( \pi a^2 \right)^2 \). Equation 8 is a computationally efficient method of calculating the viscous impedance in the frequency domain. However, it also lends itself to an implementation in the time domain. This impedance can be implemented as an analog circuit by the infinite RL ladder network shown in Figure 4. As an analog circuit, it is straightforward to implement this impedance in a time domain model using techniques that were first developed as the SPICE model for electrical circuits, and have subsequently been implemented in other computational frameworks.

**Figure 4.** The impedance \( Z d\ell \) can be implemented as this infinite network, with \( L_0 = \rho d\ell / \pi a^2 \) and \( R_v = 8\pi pv d\ell / \left( \pi a^2 \right)^2 \).

**Figure 5.** The continued fraction approximation for the impedance per unit length \( Z \) has limited bandwidth depending on the number of branches used in the approximation.

Figure 5 shows the real part of the impedance per unit length for a tube with a diameter of 1.5 cm and a length of 25 cm. The “analytical” curve is calculated using MATLAB with its full numerical precision, and the approximations are calculated using the LTspice implementation of the SPICE code. The curves for each analog circuit are labeled with the number of circuit “branches” or rungs of the ladder network, where each branch adds one resistance and one inductance to the network. Figure 4 shows a circuit with 6 branches. Figure 5 shows that the analog circuit approximation is accurate at low frequencies and diverges from the correct solution at high frequencies. The approximation is extended to a higher frequency range as more branches are added to the circuit. Using a circuit with 24 branches (49 components) gives a reasonably good approximation to more than four orders of magnitude in frequency above the transition frequency that is below 1 Hz for this tube.

Figure 6 shows the fractional error in the continued fraction approximation compared to the calculation with full numerical precision. With 24 branches, the approximation is accurate within 2% for frequencies below 10 kHz and to 0.1% below 2 kHz.
The Admittance per Unit Length $Y$

Consider next the admittance per unit length in Equation 2. The thermal function $f_t$ can be expanded as a continued fraction,$^9$ as

\[
Y \frac{d\ell}{\rho c^2} = j\omega \frac{\pi a^2}{\rho c^2} \left[ 1 + \left( \gamma - 1 \right) f_t \right] \frac{d\ell}{\rho c^2} + \frac{j\omega (\gamma - 1) \pi a^2}{\rho c^2} \frac{2 f_t \left( z_t \right)}{z_t J_0 \left( z_t \right)}
\]

\[
= \frac{j\omega \pi a^2}{\rho c^2} \frac{d\ell}{\rho c^2} + \frac{j\omega (\gamma - 1) \pi a^2}{\rho c^2} \frac{2}{z_t \frac{2}{z_t} - \frac{1}{z_t} \frac{1}{z_t} - \frac{1}{z_t} - \cdots}
\]

(9)

Using methods similar to those in the previous section, it can be shown that

\[
Y \frac{d\ell}{\rho c^2} = j\omega C_0 + \frac{1}{j\omega C_1 + \frac{1}{R_t + \frac{3}{j\omega C_1} + \frac{1}{R_t + \frac{5}{j\omega C_1} + \frac{1}{R_t + \cdots}}}}
\]

(10)

where $C_0 = \pi a^2 / \rho c^2$, $C_1 = (\gamma - 1) C_0$, and $R_t = \rho c^2 C_0 / 8\pi \kappa (\gamma - 1) d\ell$. This admittance can be implemented as the infinite RC ladder network shown in Figure 7.

FIGURE 7. The impedance $Y \frac{d\ell}{\rho c^2}$ can be implemented by this infinite network. Component values defined in the text.

Figure 8 shows the real part of the admittance per unit length for a tube with a diameter of 1.5 cm and a length of 25 cm. The “analytical” curve is calculated using MATLAB with its full numerical precision, and the approximations are calculated using the LTspice.$^{14}$ As above, the curves for the analog circuit are labeled with the number of circuit “branches” of the ladder network. Again, the approximation is extended to a higher frequency range as more branches are added to the circuit. Using a circuit with 24 branches (49 components) gives a
reasonably good approximation to more than four orders of magnitude in frequency above the adiabatic transition frequency that is below 1 Hz for this tube.

Figure 9 shows the fractional error in the continued fraction approximation compared to the calculation with full numerical precision. With 24 branches, the approximation is accurate within 2% for frequencies below 10 kHz and to 0.1% below 2.5 kHz.

**FIGURE 8.** The admittance function $Y$ is well approximated to approximately four orders of magnitude above the adiabatic transition frequency using 16 analog circuit branches.

**FIGURE 9.** The fractional error in the approximation for $Y$ depends on the number of branches used in the approximation.

### The Finite Thermoviscous Model

A complete model of the infinitesimal tube section combines the models for $Z$ and $Y$ into the impedance network of Figure 2, as shown in Figure 10. The number of branches in each leg, shown here as three, should be extended to achieve the required bandwidth, as described in the previous section. Note that the component values in the series impedance legs are changed by a factor of two from Figure 4 as indicated in Figure 2.

A tube of finite length $\ell$ can then be modeled by a number of cascaded segments of Figure 10, or by the slightly simpler configuration of Figure 11. In a practical implementation of this model, of course, only a finite number $N$ of cascaded sections can be used. The effects on the accuracy calculation of these finite approximations are shown in Figure 12. Again, the cascade approximation is accurate at low frequency. The bandwidth of accurate approximation is increased by using a greater number of shorter sections. In this case, the values of the resonance and antiresonance frequencies are within 2% when the length of the cascaded sections is 10% of a wavelength. This resolution must be considered in every particular application.
FIGURE 10. The truncated model of the short tube segment. Each of the three ladder networks should be extended with enough branches to obtain the required bandwidth.

FIGURE 11. The complete model for a tube of finite length including thermoviscous losses.

FIGURE 12. Acoustic input impedance of the tube. Green curve is the analytic expression. Blue curve uses the marked number of segments as shown in Figure 11.
EXAMPLE: HEARING AID SOUND TUBE

One possible application for the methods described here is to model sound propagation in the small tubes that are used to couple sound from a hearing aid to the ear canal of the user. A common size for this tubing is 1 mm inside diameter, in whatever length is needed for the application. A 5 cm total length, longer than most actual applications, will be used here. Figure 13 shows the analytically calculated acoustic input impedance compared with the analog circuit model using 10 and 20 segments. Those using this model would need to decide whether are necessary for an adequate approximation at the upper end of the desired band.

FIGURE 13. Acoustic input impedance of a tube with 1 mm diameter and 5 cm length, showing the effect of increasing the number of segments in the model.

EXAMPLE: A CLARINET-LIKE AIR COLUMN

As an example of the use of this tube model, consider an air column whose dimensions are approximately those of a clarinet. The inside diameter of the tube is 1.5 cm and the length is 25 cm. This tube was analyzed earlier with its far end terminated in a zero pressure condition, with the input impedance shown in Figure 12. For the present example, the tube is terminated in a lattice of open tone holes that provide a 1500 Hz lattice cutoff frequency. An infinite lattice is approximated by ten open open tone holes separated by ten tube segments. The open end is terminated in a pure resistance equal to the characteristic impedance of the tube. Each segment of the tube is modeled using the analog circuit approximation described in this paper, with shorter segments requiring fewer cascaded sections. Note that the spacing of the lattice is not intended to provide the notes of a musical scale, only to provide a strong cutoff behavior as would be present in an actual woodwind bore. The acoustic input impedance of this clarinet-like tube is shown in Figure 14. When compared with the similar curves of Figure 12, this clearly shows the effect of the tone hole lattice cutoff frequency at 1500 Hz. The resonances in the vicinity of 8 kHz are modes of the system that have pressure nodes near the open tone holes, and are thus not significantly affected by the lattice. Also for these modes, the terminating resistance is relatively ineffective because, as Benade points out, the characteristic impedance of the tone hole lattice is frequency dependent. In a more realistic clarinet model, the spacing and sizes of the tone holes would change along the length of the bore, and probably reduce the amplitudes of the peaks near 8 kHz.

FIGURE 14. Acoustic input impedance for a clarinet-like tube having 1.5 cm inside diameter, 25 cm closed length with ten segments of a tone hold lattice designed to give a 1500 Hz cutoff frequency.
CONCLUSION

The approximation presented in this paper is able to provide arbitrarily good precision over any defined finite bandwidth, within the limits of the numerical accuracy of the machine that is employed for the computation. The continued fraction computation is generally robust in a numerical computation as it does not result in the subtraction of approximately equal numbers. The apparent disadvantage when this is implemented in an analog circuit computation is the large number of components required to model a single tube. For example, the tube in the third graph in Figure 12 includes approximately 4750 components and a matrix size to solve at each frequency, or each time step, of 2424. While not insignificant, this is not an especially large problem size for modern circuit analysis programs.

REFERENCES

10. Information on SPICE can be found at http://bwrc.eecs.berkeley.edu/Classes/IcBook/SPICE/ (date last viewed June 26, 2012).
14. The LTspice program is freely available from Linear Technology Corp. through their web site at http://www.linear.com (date last viewed June 26, 2012).