Numerical mode-matching approach for acoustic attenuation prediction of expansion chambers with single inlet and double outlets

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Numerical mode matching (NMM) method is developed to predict the acoustic attenuation performance of expansion chambers with single-inlet and double-outlets. The two-dimensional finite element method is employed to calculate the transversal eigenvalues and eigenvectors, and the mode matching technique is used to determine the modal amplitudes and transmission loss of expansion chamber silencers by combining the boundary conditions at inlet and outlets. For the purpose of validation, the transmission loss predictions of elliptical expansion chambers with single-inlet and double-outlets from the present NMM method and the three-dimensional finite element method (FEM) are compared, and good agreements between them are observed. Then the NMM method is used to investigate the effects of extended lengths of outlets on the acoustic attenuation performance of elliptical expansion chambers.
INTRODUCTION

The expansion chamber with extended inlet/outlet exhibits a desirable acoustic attenuation performance as a combination of the broad band domes of simple expansion chamber and the resonant peaks of quarter-wave resonators. Selamet and Ji [1] developed an analytical mode-matching approach to investigate the effects of length of expansion chamber and extensions of inlet and outlet into the chamber on the multi-dimensional wave propagation and acoustic attenuation performance of the concentric expansion chambers, their calculated results of transmission loss agree well with the experiments and the boundary element method (BEM) predictions. These works, however, have chosen not to elaborate on the effects of offset extended inlet and outlet ducts on the acoustic attenuation performance of these configurations. Selamet et al. [2-4] discussed the effects of offset inlet and outlet on the acoustic attenuation performance of circular expansion chambers by using the analytical mode-matching approach, boundary element method and experiment. Denia et al. [5] and Selamet et al. [6] gave simple polynomial expressions to evaluate the plane wave cut-off frequency and to find the optimum location of the outlet tube for elliptic chamber silencers by means of Mathieu functions, and then the influence of chamber length and inlet/outlet location on the acoustic attenuation performance was studied by using the modal superposition method and the point source technique. However, the studies mentioned above [2-6] excluded the extensions of inlet and outlet(s) into the chamber, which is widely used configuration in the practical silencer designs.

The expansion chamber with single-inlet and double-outlets has been used a silencer in the exhaust system of internal combustion engine, since it may provide desirable broadband acoustic attenuation and low pressure drop. As the analytical mode matching method is not applicable for the expansion chambers with extended single-inlet and double-outlets as the eigenfunction of this configuration may not be obtained analytically, the present work develops a numerical mode matching method to predict the acoustic attenuation performance of the elliptical expansion chamber with extended single-inlet and double-outlets. The numerical mode matching method can be used to predict the acoustic attenuation performance of the silencer accurately as a three-dimensional analytical approach, and also can be used to analyze the effect of the higher order modes directly. The numerical mode-matching method has been reported first by Kirby [7] to calculate the acoustic attenuation performance of the dissipative perforated tube silencers of arbitrary but uniform across-section with mean flow. The transmission loss predictions form the numerical mode-matching method for two elliptical concentric dissipative perforated tube silencers with mean flow agree well with the experimental measurements in the frequency range below 1500Hz.

The objective of the present study is then to develop a numerical mode-matching approach to predict the acoustic attenuation performance of expansion chamber with arbitrary shaped cross-section and extended inlet/outlets. The 2-D finite element method is employed to determine the transversal modes and corresponding wavenumbers of the cross-sections, and the mode-matching technique is utilized to establish the relationship among the modal amplitudes by combining the continuity and boundary conditions as well as by means of the orthogonality of eigenfunctions, and then the acoustic properties may be obtained for the given boundary conditions at the inlet and outlets. The present numerical mode-matching approach is validated by comparing the transmission loss results from 3-D FEM for an elliptical expansion chamber with extended single-inlet and double-outlets. Finally, the effect of extended length of outlets on the acoustic attenuation performance of elliptical expansion chamber is investigated.

THEORY

The harmonic sound field inside the silencer is governed by Helmholtz equation

\[ \nabla^2 p(x, y, z) + k^2 p(x, y, z) = 0 \]  

where, \( \nabla^2 \) is the Laplacian operator in the Cartesian coordinate system, \( p \) is the sound pressure, \( k = 2\pi f / c_0 \) is the wavenumber, \( f \) is the frequency and \( c_0 \) is the speed of sound.
For the expansion chamber with uniform cross-section, the sound pressure may be expressed as

\[ p(x, y, z) = \phi(x, y)Z(z) \]  

(2)

where \( \phi(x, y) \) represents the transversal sound pressure components. Considering the rigid wall, the corresponding transversal boundary condition may be expressed as

\[ \frac{\partial \phi(x, y)}{\partial n} = 0 \]  

(3)

Substituting Eq. (2) into Eq. (1) yields

\[ \frac{1}{\phi(x, y)} \nabla^2 \phi(x, y) + \frac{1}{Z(z)} \frac{d^2Z(z)}{dz^2} + k^2 = 0 \]  

(4)

where \( \nabla^2 \) is the Laplacian operator in the two-dimensional Cartesian coordinate system. Eq. (4) may be divided into the following two independent equations

\[ \nabla^2 \phi(x, y) + k_{xy}^2 \psi(x, y) = 0 \]  

(5)

\[ \frac{d^2Z(z)}{dz^2} + k_z^2 Z(z) = 0 \]  

(6)

where \( k_{xy} \) and \( k_z \) are the wavenumbers in the transversal and axial directions, respectively, and related with the compatibility condition

\[ k_{xy}^2 + k_z^2 = k^2 \]  

(7)

For an arbitrary shaped cross-section, Eq. (5) may be solved numerically, such as the two-dimensional FEM, and theoretically there are infinite solutions of Eq. (5). The general solution of Eq. (6) may be expressed as

\[ Z(z) = C_1e^{-jk_zz} + C_2e^{jk_zz} \]  

(8)

Therefore, the sound pressure may be expressed as

\[ p(x, y, z) = \sum_{i=0}^{\infty} \phi(x, y)(C_1e^{-jk_{xy}z} + C_2e^{jk_{xy}z}) \]  

(9)

where \( i \) represents the order of transversal modes.

In order to obtain the transversal sound pressure components, the 2-D finite element method is used to solve Eq. (5). The transversal sound pressure components are expressed as

\[ \phi(x, y) = N^T \mathbf{p} \]  

(10)

where \( N \) is the column vector of global shape functions, \( \mathbf{p} \) is column vector of nodal transversal sound pressure components.

Combining the boundary condition (3) and applying the method of Galerkin weighted residuals to Eq. (5) yield the following eigenequation for the cross-section

\[ (\mathbf{K} - k_{xy}^2 \mathbf{M}) \mathbf{p} = 0 \]  

(11)

where, \( \mathbf{K} = \sum_{e} \int_{S_e} (\nabla N)_e (\nabla N)_e^T dS_e \) and \( \mathbf{M} = \sum_{e} \int_{S_e} (N)_e (N)_e^T dS_e \) are the global stiffness matrix and mass matrix in the cross-section, respectively, subscript "e" denotes the element.

Supposing number of nodes is \( n \) in the cross-section, the infinite solutions \( \phi(x, y) \) of Eq. (5) may be truncated to the finite \( n \), and then \( n \) eigenvalues \( k_{xyi} \) and associated eigenvectors \( \{ \Phi_{xyi} \} \) with length of \( n \) may be obtained by solving Eq. (11). Combining all the eigenvalues and eigenvectors composes the vector \( \mathbf{k}_{xy} \) and the matrix \( \Phi_{xy} \), where the length of \( \mathbf{k}_{xy} \) is \( n \) and the dimension of \( \Phi_{xy} \) is \( n \times n \). The vector of nodal transversal sound pressure components may be expressed as

\[ \mathbf{p} = \sum_{i=0}^{n} \beta_i \{ \Phi_{xyi} \} = \Phi_{xy}^T \mathbf{\beta}, \]  

where \( \mathbf{\beta} \) represents the vector of modal coefficients with length \( n \).
Then the transversal sound pressure components may be rewritten as

$$
\Phi(x, y) = N^T \Phi_{xy}^T \beta = \sum_{i=0}^{n} \beta_i N^T \left( \Phi_{xy}^i \right)
$$

(12)

where \( i \) denotes the modal order, \( \beta_i \) and \( \left( \Phi_{xy}^i \right) \) represent the modal coefficient and transversal eigenvector corresponding to the \( i \)-th order mode, respectively.

Therefore, the sound pressure may be expressed as

$$
p(x, y, z) = \sum_{i=0}^{n} N^T \left( \Phi_{xy}^i \right) \beta_i (C_1 e^{-jk_{xy}z} + C_2 e^{jk_{xy}z}) = \sum_{i=0}^{n} \Psi_i(x, y) (C_1' e^{-jk_{xy}z} + C_2' e^{jk_{xy}z})
$$

(13)

where \( \Psi_i(x, y) = N^T \left( \Phi_{xy}^i \right) \).

**FIGURE 1.** Elliptical expansion chamber with extended single-inlet and double-outlet

The elliptical expansion chamber with extended single-inlet and double-outlets shown in Fig. 1 is selected as an example to describe the numerical mode matching approach. The silencer may be divided into seven acoustic regions by the extended tubes: \( V_A, V_B, V_C, V_D, V_E, V_F \) and \( V_G \). The corresponding cross-sections are denoted by \( S_A, S_B, S_C, S_D, S_E, S_F \) and \( S_G \), respectively, where, \( S_B = S_C + S_D, S_D = S_C - S_E \) and \( S_F = S_D - S_G \). The respective transversal wavenumbers and eigenvectors in the seven cross-sections may obtained by solving the eigenequation (11).

In order to apply the 2-D FEM to obtain the transversal wavenumbers and eigenvectors, all finite element meshes for the cross-sections of inlet duct, annular chambers and outlet ducts have to map onto the cross-section of expansion chamber.

**MODE-MATCHING APPROACH**

Considering the expansion chamber shown in Fig. 1, the sound pressures and axial particle velocities in the seven regions may be expressed as

$$
p_{ij}(x, y, z) = \sum_{i=0}^{n} \Psi_{hi}(x, y) \left( I_{i} e^{-jk_{xy}z} + I_{i} e^{jk_{xy}z} \right)
$$

(14)

$$
U_{I,j}(x, y, z) = \frac{1}{\rho_0 \omega} \sum_{i=0}^{n} \Psi_{hi}(x, y) k_{zI} \left( I_{i} e^{-jk_{xy}z} - I_{i} e^{jk_{xy}z} \right)
$$

(15)

where, \( I = A, B, C, D, E, F, G \), \( z_j = \begin{cases} z - L_C & I = D, E \ , \ L_C = L - L_2 \ , \ L'_C = L - L_1 - L_3 \ . \\ z - L'_C & I = F, G \end{cases} \)

\( \Psi_{hi}(x, y) = N^T \left( \Phi_{xy}^i \right) \) represents the \( i \)-th order transversal eigenfunction for the region \( I \), \( I' \) and \( I'' \) are the...
modal amplitudes of traveling waves in the positive and negative z directions in the seven regions, respectively,
\[
k_{ti} = \begin{cases} \sqrt{k^2 - k_{toi}^2} & (k^2 > k_{toi}^2) \\ -j\sqrt{k^2 - k_{toi}^2} & (k^2 < k_{toi}^2) \end{cases}
\]
is the \(i\)-th order axial wavenumber in the region \(I\).

The continuity conditions of sound pressure and particle velocity at the inlet and outlet cross-sections may be written as
\[
p_C(z = 0) = p_A(z = 0), \quad (x, y) \in S_A \tag{16}
p_C(z = 0) = p_B(z = 0), \quad (x, y) \in S_B \tag{17}
U_{C,2}(z = 0) = \begin{cases} U_{A,2}(z = 0), \quad (x, y) \in S_A \\ U_{B,2}(z = 0), \quad (x, y) \in S_B \end{cases} \tag{18}
p_C(z = L_C) = p_D(z = L_C), \quad (x, y) \in S_D \tag{19}
p_C(z = L_C) = p_E(z = L_C), \quad (x, y) \in S_E \tag{20}
U_{C,2}(z = L_C) = \begin{cases} U_{D,2}(z = L_C), \quad (x, y) \in S_D \\ U_{E,2}(z = L_C), \quad (x, y) \in S_E \end{cases} \tag{21}
p_D(z = L_C') = p_F(z = L_C'), \quad (x, y) \in S_F \tag{22}
p_D(z = L_C') = p_G(z = L_C'), \quad (x, y) \in S_G \tag{23}
U_{D,2}(z = L_C') = \begin{cases} U_{F,2}(z = L_C'), \quad (x, y) \in S_F \\ U_{G,2}(z = L_C'), \quad (x, y) \in S_G \end{cases} \tag{24}

The rigid wall boundary conditions at the left and right endplates may be expressed as
\[
U_{B,2}(z = -L_a) = 0, \quad (x, y) \in S_B \tag{25}
U_{F,2}(z = L_c + L_a) = 0, \quad (x, y) \in S_F \tag{26}
\]

Substituting the expressions of the sound pressures and particle velocities into the continuity conditions, and multiplying the continuity conditions by the eigenfunctions of the integration cross-sections and then integrating yields
\[
\sum_{i=0}^{n} (C_i^+ + C_i^-) \left\langle \Psi_{CI} | \Psi_{AI} \right\rangle_{S_A} = \sum_{i=0}^{n} (A_i^+ + A_i^-) \left\langle \Psi_{AI} | \Psi_{AI} \right\rangle_{S_A} \quad j = 0,1,2...n \tag{27}
\]
\[
\sum_{i=0}^{n} (C_i^+ + C_i^-) \left\langle \Psi_{CI} | \Psi_{BI} \right\rangle_{S_B} = \sum_{i=0}^{n} (B_i^+ + B_i^-) \left\langle \Psi_{BI} | \Psi_{BI} \right\rangle_{S_B} \quad j = 0,1,2...n \tag{28}
\]
\[
\sum_{i=0}^{n} (C_i^+ - C_i^-) k_{Cz} \left\langle \Psi_{CI} | \Psi_{CJ} \right\rangle_{S_C}
\]
\[
= \sum_{i=0}^{n} (A_i^+ - A_i^-) k_{A2l} \left\langle \Psi_{AI} | \Psi_{CI} \right\rangle_{S_A} + \sum_{i=0}^{n} (B_i^+ - B_i^-) k_{B2l} \left\langle \Psi_{BI} | \Psi_{CI} \right\rangle_{S_B} \quad j = 0,1,2...n \tag{29}
\]
\[
\sum_{i=0}^{n} (C_i^e^{-j\omega t} + C_i^-) \left\langle \Psi_{CI} | \Psi_{Dj} \right\rangle_{S_D} = \sum_{i=0}^{n} (D_i^+ + D_i^-) \left\langle \Psi_{DI} | \Psi_{DI} \right\rangle_{S_D} \quad j = 0,1,2...n \tag{30}
\]
\[
\sum_{i=0}^{n} (C_i^e^{-j\omega t} + C_i^-) \left\langle \Psi_{CI} | \Psi_{Ej} \right\rangle_{S_E} = \sum_{i=0}^{n} (E_i^+ + E_i^-) \left\langle \Psi_{EI} | \Psi_{EI} \right\rangle_{S_E} \quad j = 0,1,2...n \tag{31}
\]
\[
\sum_{i=0}^{n} (C_i e^{-j\lambda_{\text{in}} l_{\text{in}}} - C_i e^{j\lambda_{\text{in}} l_{\text{in}}}) k_{\text{CI}} \left\langle \psi_{C_i} \psi_{C_j} \right\rangle_{S_e} = \sum_{i=0}^{n} (D_i - D_i) k_{\text{DI}} \left\langle \psi_{D_i} \psi_{D_j} \right\rangle_{S_e} + \sum_{i=0}^{n} (E_i - E_i) k_{\text{EI}} \left\langle \psi_{E_i} \psi_{E_j} \right\rangle_{S_e} \quad j = 0, 1, 2... n (32)\]

\[
\sum_{i=0}^{n} (D_i e^{-j\lambda_{\text{in}} (l_{\text{in}} - l_{\text{I}_j})} + D_i e^{j\lambda_{\text{in}} (l_{\text{in}} - l_{\text{I}_j})}) \left\langle \psi_{D_i} \psi_{I_j} \right\rangle_{S_e} = \sum_{i=0}^{n} (F_i^+ + F_i^-) \left\langle \psi_{F_i} \psi_{F_j} \right\rangle_{S_e} \quad j = 0, 1, 2... n (33)\]

\[
\sum_{i=0}^{n} (D_i e^{-j\lambda_{\text{in}} (l_{\text{out}} - l_{\text{I}_j})} + D_i e^{j\lambda_{\text{in}} (l_{\text{out}} - l_{\text{I}_j})}) \left\langle \psi_{D_i} \psi_{G_j} \right\rangle_{S_e} = \sum_{i=0}^{n} (G_i^+ + G_i^-) \left\langle \psi_{G_i} \psi_{G_j} \right\rangle_{S_e} \quad j = 0, 1, 2... n (34)\]

\[
\sum_{i=0}^{n} (D_i e^{-j\lambda_{\text{in}} (l_{\text{out}} - l_{\text{I}_j})} - D_i e^{j\lambda_{\text{in}} (l_{\text{out}} - l_{\text{I}_j})}) k_{\text{DI}} \left\langle \psi_{D_i} \psi_{D_j} \right\rangle_{S_e} = \sum_{i=0}^{n} (F_i^+ - F_i^-) k_{\text{FI}} \left\langle \psi_{F_i} \psi_{F_j} \right\rangle_{S_e} + \sum_{i=0}^{n} (G_i^+ - G_i^-) k_{\text{GI}} \left\langle \psi_{G_i} \psi_{G_j} \right\rangle_{S_e} \quad j = 0, 1, 2... n (35)\]

By the orthogonality of eigenfunctions, the following relations may be obtained from Eq. (25) and Eq. (26)

\[
B_i^+ = B_i^e e^{-2j\theta_{\text{in}} l_{\text{in}}} (36)\]

\[
F_i^- = F_i^e e^{-2j\theta_{\text{out}} l_{\text{out}}} (37)\]

And the integrations in equations (27)-(35) may be expressed as

\[
\left\langle \psi_{F_i} \psi_{F_j} \right\rangle_{S_e} = \int S (\psi_{F_i}(x, y) \psi_{F_j}(x, y)) dS = \int \left( N^T(\Phi_{F_i}) \right) \left( N^T(\Phi_{F_j}) \right) dS = \int \left( \Phi_{F_i}^T \right) N N^T \left( \Phi_{F_j} \right) dS = \left( \Phi_{F_i} \right)^T M \left( \Phi_{F_j} \right) = \begin{cases} 0, & i \neq j \\ \| \Phi_{F_i} \|^2, & i = j \end{cases} (38)\]

In equations (27)-(37), there are 14 \( (n+1) \) unknown modal amplitudes, supposing that the incoming wave in the inlet duct is planar (setting \( A_i^+ = 1; A_i^- = 0, i > 0 \) ) and anechoic terminations are imposed at the exits of the chamber (setting \( E_i^- = 0 \) and \( G_i^- = 0 \)), thus number of unknowns turns to 11 \( (n+1) \). Since higher modes have a diminishing effect on the solution, \( n \) can be truncated to \( N \) based on the computational frequency and the size of silencer. Thereby the 11 \( (N+1) \) unknown modal amplitudes can be obtained by solving the 11 \( (N+1) \) equations. For the geometries and frequency range considered in the present study, \( N = 25 \) were found to be sufficient. Once equations (27)-(37) are solved, the transmission loss of silencer is determined by

\[
TL = -10 \log_{10} \left[ \left( E_0^+ \right)^2 S_e + \left( G_0^- \right)^2 S_G \right] (39)\]

if the frequency is below the plane wave cut-off frequency of the outlet ducts.

**RESULTS AND DISCUSSION**

For the elliptical expansion chamber with extended single-inlet and double-outlets shown in Fig. 1, the present study considers \( L = 0.2823 m \) for the length of chamber, \( a = 0.2043 m \) and \( b = 0.1149 m \) for the major and minor axes of chamber, \( d_i = d_3 = d_3 = 0.0486 m \) for the diameters of inlet and outlet ducts, \( \delta = 0.06 m \) for the offset of outlet. \( L_1, L_2, \) and \( L_3 \) represent the extensions of inlet and outlet ducts into the chamber, respectively.

Figures 2-3 show the transmission loss predictions from the NMM and 3-D FEM for the expansion chamber with different outlet extensions. Excellent agreements between the NMM and FEM predictions demonstrated the accuracy of the numerical mode matching method in predicting the transmission loss of expansion chamber with
extended single-inlet and double-outputs. In order to examine the effects of higher order modes on the transmission loss of the expansion chamber, the corrected 1-D analytical solutions are included also in Figs. 2-3.

**FIGURE 2.** Transmission loss of elliptical expansion chamber with extended single-inlet and double-outlet; \( L_1 = 0.08m \), \( L_2 = 0.06m \) and \( L_3 = 0.04m \) (— numerical mode-matching approach; —— finite element method; -----corrected 1-D analytical method)

**FIGURE 3.** Transmission loss of elliptical expansion chamber with extended single-inlet and double-outlet; \( L_1 = 0.08m \) and \( L_2 = L_3 = 0.06m \) (— numerical mode-matching approach; —— finite element method; -----corrected 1-D analytical method)

The cut-off frequency of plane wave in the inlet and outlet ducts is determined as 4148Hz. For the elliptical expansion chamber considered in Figs.2-3, there are four representing cross-sections. Table 1 gives the first four modal shapes and corresponding modal frequencies for the four representing cross-sections.

As the inlet locates on the nodal lines of the first two modes of \( S_1 \) and \( S_2 \), so the first two modes cannot be excited. For the silencer with different extended lengths of outlets, the outlets locate on the major axes of \( S_1 \), and the effect of 1st mode may not be eliminated since the asymmetry, however because that the length of the chamber corresponding to \( S_1 \) is short, so the effect of the modes of \( S_1 \) is negligible. For the silencer with the same extended lengths of outlets, the effects of the first two modes of \( S_1 \) may be eliminated due to the symmetry. Therefore the plane wave cut-off frequencies of the expansion chamber considered in Figs.2-3 both extend to the 3rd modal frequency (1451Hz) of \( S_4 \) as indicated in the transmission loss curves. Above the cut-off frequency, the deviation between the three-dimensional results and the corrected one-dimensional analytical solutions are observed. Comparing the two transmission loss curves, it can been seen that there is only one resonance peak determined by the inlet extension below the plane wave cut-off frequency in Fig.2, however two resonance peaks corresponding to inlet extension and outlet extension are observed in Fig.3. The transmission loss curves indicate that the resonance peak may be produced when the extended lengths of two outlets are same only. It is explained as that two different extended lengths of outlets may produce two different frequency side-branch resonances, and the acoustic energy is
reflected mostly in one outlet and is transmitted in another outlet, therefore the resonance peak may not appear in the transmission loss curve.

**TABLE 1.** Transversal modes and modal cut-on frequencies of cross-sections of elliptical expansion chamber with extended single-inlet and double-outlets

<table>
<thead>
<tr>
<th>Mode</th>
<th>$S_1$</th>
<th>$S_2$</th>
<th>$S_3$</th>
<th>$S_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st order</td>
<td>1002 Hz</td>
<td>847 Hz</td>
<td>1016 Hz</td>
<td>1025 Hz</td>
</tr>
<tr>
<td>2nd order</td>
<td>1699 Hz</td>
<td>1416 Hz</td>
<td>1481 Hz</td>
<td>1371 Hz</td>
</tr>
<tr>
<td>3rd order</td>
<td>1822 Hz</td>
<td>1882 Hz</td>
<td>1598 Hz</td>
<td>1451 Hz</td>
</tr>
<tr>
<td>4th order</td>
<td>2297 Hz</td>
<td>2212 Hz</td>
<td>2087 Hz</td>
<td>1824 Hz</td>
</tr>
</tbody>
</table>

**CONCLUSIONS**

A numerical mode-matching approach is developed to predict the acoustic attenuation performance of expansion chambers with extended single-inlet and double-outlets. For the elliptical expansion chambers, the transmission loss predictions from the present NMM agree well with the results from the 3-D FEM, which verified the accuracy of the present NMM approach. The effect of the extended lengths of outlets on the acoustic attenuation performance is investigated. The studies show that the side-branch resonance may be produced by outlet extensions only when the two extended lengths of the double-outlets are same in the frequency range of plane wave.

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