2aPAa1. Bandgap formation mechanisms in periodic materials and structures
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Bandgaps are frequency intervals of no wave propagation for mechanical waves. Spatial periodicity provides one mechanism for the emergence of bandgaps in periodic composite materials such as lattice materials, phononic crystals, and acoustic metamaterials. Coupling a propagating wave in a periodic medium with a local resonator provides an alternate mechanism for bandgap emergence. This study examines these two band-gap formation mechanisms using a receptance coupling technique. The receptance coupling technique yields closed-form expressions for the location of bandgaps and their width. Numerical simulations of Blochwaves are presented for the Bragg and sub-Bragg bandgaps and compared with the bounding frequency predictions given by the receptance analysis of the unitcell dynamics. It is observed that the introduction of periodic local resonators narrows Bragg bandgaps above the local resonant bandgap. Introduction of two fold periodicity is shown to widen the Bragg bandgap, thus expanding the design space. The generality of the receptance technique presented here allows straightforward extension to higher dimensional systems with multiple degrees of freedom coupling. Implication of this study for Nano-Electro-Mechanical Systems (NEMS) will be discussed.

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INTRODUCTION

Bandgaps are frequency intervals in which wave propagation is forbidden in periodic materials [1, 2, 3, 4] over a wide range of length scales: from the phonon spectrum of a sandwiched single layer Graphene at nanoscale [5] to dispersion band structure of multi-phase periodic composite materials [3, 6] at mesoscale and periodic structures [7, 8] at even larger length scales. Two broad mechanisms have been identified: Bragg scattering and local resonance. In his study on light wave propagation through periodic gratings [1, 2], Bragg had shown that bandgaps appear at wavelengths on the order of the spatial periodicity or size of unitcell. Bragg bandgaps require a prohibitively large unitcell size to achieve bandgaps at long wavelengths. The quest to overcome this size limitation lead to the discovery of a local resonant mechanism [4] resulting in sub-Bragg bandgaps in metamaterials. Superior acoustic transmission loss at low frequencies — not possible using conventional materials — was achieved using spatially compact metamaterials. Rapid advances in micro fabrication technologies [9] have revived interest in materials with periodic micro-architecture [10] and a renewed interest in periodic structure theory [7, 11, 8, 12, 13]. It has been established [14, 7] that for a symmetric unitcell the edges of the passband are bound by the so called "locked or fixed" and "free" natural frequencies of the unitcell. A symmetric unitcell is defined to be the one with a reflective symmetry about its mid-plane so that the left and right edges are indistinguishable. For asymmetric unitcells it is not guaranteed that the band edges are bound by the resonant frequencies of the unitcells [14].

Design of novel material topologies requires efficient methods to compute bandgap widths — regardless of their physical origins — and the dependence of bandgap widths on material and geometric parameters. Towards this end, a receptance coupling technique will be developed to predict the bounding frequencies of bandgaps. Receptance functions, or Frequency Response Functions (FRFs), have many computational advantages; they can be measured using standard modal testing procedures [15], which can be adapted to the length scale of interest. Design guidelines for local resonators can be readily obtained from this technique. Moreover, the receptance approach can be extended to higher dimensions and multi degree coupling, avoiding the solution of complex partial differential equations governing the wave propagation in the underlying medium. All the above motivate this study with a focus on material and structural systems with a symmetric unitcell. Illustrations are provided using a structural example, though the same procedure can be applied to other length scales as well.

BANDGAPS FROM RECEPTANCE COUPLING TECHNIQUES

Receptance is defined as the steady harmonic displacement response of a linear system subjected to a harmonic input force of unit amplitude. Receptance functions depend on the forcing frequency and the spatial location of the force and response points. Of particular interest here are point receptance functions for which the force and response points coincide. Inverse Fourier transform of receptance leads to impulse response function in the time domain [16, 15]. Simple rules exist for coupling point receptances of two systems, each evaluated at the point of coupling. For two systems with point receptances $H_1$ and $H_2$, the following results can be derived:

\[
\begin{align*}
\frac{1}{H} &= \frac{1}{H_1} + \frac{1}{H_2}, & \text{parallel connection} \\
H &= H_1 + H_2, & \text{series connection}
\end{align*}
\]  

(1)

The above rules provide the characteristic equations whose roots are the natural frequencies of the coupled system. Taking $H_1$ and $H_2$ as the point receptance functions of the unitcell comprising a wave guide (such as a beam) and a local resonator, one can calculate the natural frequencies of the unitcell. Hence the bounding frequencies of bandgaps and bandgap widths of the infinite periodic system produced by repeating the unitcell.

Consider the bandgaps of three periodic beam systems, shown in Fig. 1, each obtained by truncating an infinite periodic system. The general case of two-fold periodicity of the beam with periodic arrangement of masses and damped resonators will now be studied in detail. Point receptance of a beam at any location, $x$,
can be expressed in terms of the normal mode-shapes of the beam using the modal series expansion [16, 15] as:

\[ H_b = \sum_{r=1}^{n} \frac{\Phi_r^2(x)}{a_r(\omega_r^2 - \omega^2 + i2\zeta_r\omega_\omega r)}, \quad i = \sqrt{-1}, \]  

(2)

where \( \Phi_r(x) \) is the modeshape associated with the normal mode \( r \) with natural frequency \( \omega_r \) and \( \zeta_r \) is the associated damping ratio. Mass normalization constant is \( a_r \). For the unitcells shown in Fig. 2, \( x = \frac{1}{2} \) where \( l \) is the length of the unitcell. Point receptance of the damped resonator at the point of attachment can be shown to be:

\[ H_a = -\frac{k_a - m_a\omega^2 + ic_a\omega}{(k_a + ic_a\omega)m_a\omega^2}, \]  

(3)

where, \( k_a \), \( m_a \), and \( c_a \) correspond to the stiffness, mass and damping coefficient of the resonator respectively. The receptance of the lumped mass \( M \) is given as:

\[ H_m = -\frac{1}{M\omega^2}. \]  

(4)

We can now use the systems in parallel coupling rule from eq.(1), since all of the sub-systems (mass, resonator, and the beam) have the same displacement at the point of coupling:

\[ \frac{1}{H} = \frac{1}{H_b} + \frac{1}{H_a} + \frac{1}{H_m}. \]  

(5)

**Figure 1:** (a) Timoshenko beam with periodic masses; (b) Timoshenko beam with periodic local resonators; (c) Timoshenko beam with two-fold periodicity.

Substituting eq.(2), eq.(3) and eq.(4) in eq.(5) and solving the resulting characteristic equation yields the natural frequencies of the unitcells shown in Fig. 2 for the locked and free boundary conditions, respectively. It must be pointed out that not all terms in the series expansion of the receptance in eq.(2) need be included for the purposes of computing natural frequencies of the unitcells. For well-separated natural modes of the beam, a local approximation around each mode is adequate. First bandgap formation will be illustrated in
the following analysis. Now, in order to determine the natural frequency of the simply supported unitcell as shown in Fig. 2(a), approximate the receptance around the first mode, \( r = 1 \), as:

\[
H_b \approx \frac{\Phi_2^0(x)}{a_1(\omega_b^2 - \omega^2 + i2\zeta_1 \omega \omega_b)}, \quad i = \sqrt{-1},
\]

(6)

where \( \Phi_b(x) = \sin \frac{\pi x}{l} \) is the first mode-shape of the pinned-pinned beam and \( \omega_b = \sqrt{\frac{E}{\rho AL}} \), where \( E \) is the Young’s modulus of the beam material; \( A \) is area of cross-section; \( I \) is second moment of area with respect to neutral axis. The normalization constant for unit modal mass is \( a_1 = \frac{\rho AL}{2} \). Substituting the receptances in eq.(6), eq.(3) and eq.(4) in eq.(5) and simplifying the resulting expression gives the characteristic equation governing the natural frequencies of the coupled beam with periodic masses and resonators:

\[
-2(k_a - m_a \omega^2 + ic_a \omega)M \omega^2 + (k_a - m_a \omega^2 + ic_a \omega)pAL(\omega_b^2 - \omega^2 + i2\zeta_1 \omega \omega_b) - 2m_a \omega^2(k_a + ic_a \omega) = 0.
\]

(7)

Ignoring damping in the beam \( \zeta_b = 0 \), the above can be solved for one of the bounding frequencies of the simply supported beam with periodic masses and resonators as:

\[
\omega_{ss} = \omega_a \frac{(2p + 1 + \omega_r^2 + 2m_r)}{4p + 2} \pm \sqrt{\Lambda}
\]

with 
\[
\Lambda = (2p + 1 + \omega_r^2 + 2m_r)^2 - 4(2p + 1)\omega_r^2
\]

(8)

(9)

where \( \omega_{ss} \) stands for the natural frequency of the beam with simply supported end condition; \( p \) is the ratio of the mass \( M \) to the mass of the beam \( m_a \); \( \omega_r \) is the ratio of the frequency of the beam to that of the resonator and \( m_r \) is the ratio of the mass \( m_a \) of the resonator to that of the beam. Repeating a similar receptance analysis for higher modes of the simply supported beam \( r = 2, 3, \ldots \) gives bounding frequencies for higher bandgaps.

Receptance analysis of the guided beam, shown in Fig. 2(b) will give the second bound for the band gap. It is important to recognize that the coupling spring of the resonator gives finite potential energy for any motion of the beam, including the rigid-body motion. The one term series approximation of the receptance function for the guided beam for zero frequency rigid-body mode is:

\[
H_b = \frac{1}{\rho AL(\omega^2)},
\]

(10)

Following the same procedure used earlier, the characteristic equation is:

\[
(k_a - m_a \omega^2 + ic_a \omega)M \omega^2 + (k_a - m_a \omega^2 + ic_a \omega)pAL \omega^2 + m_a \omega^2(k_a + ic_a \omega) = 0
\]

(11)

For an undamped case, the non-zero natural frequency is:

\[
\omega_{\text{guided}} = \omega_a \sqrt{\frac{1 + \frac{m_r}{1 + p}}}
\]

(12)
Thus, the natural frequency of the unitcell of the system is once again dependent on the frequency of the resonator. The width of the first sub-Bragg bandgap is the difference between the natural frequencies of the guided and simply supported unitcells and is given by:

$$
\delta \omega = \omega_a \left[ \sqrt{1 + \frac{m_r}{1+p}} - \frac{(2p + 1 + \omega_r^2 + 2m_r) \pm \sqrt{\Delta}}{4p + 2} \right].
$$

(13)

Similar expressions for the first band gap width may be derived for the damped case.

**Blochwave Analysis**

The band structure of the infinite versions of the finite systems, shown in Fig. 1, can be numerically computed using a finite element model of the beam and Bloch analysis, see [6] for the formulation. Similarly, the receptance functions of a finite periodic beam can be numerically calculated using the finite element model by applying a force input near the left end and measuring displacement response measured at the right end of the beam. Dispersion curves can be computed by solving the following Bloch wave eigenvalue problem:

$$
\tilde{K}q = \omega^2 \tilde{M}q, \quad \tilde{K} = T^H KT, \quad \tilde{M} = T^H MT
$$

(14)

where $T$ is the wavenumber dependent Bloch transformation matrix [6] used to reduce the stiffness matrix $K$ and the mass matrix $M$ of the unitcell assembled using finite element procedures. Hermitian transpose operator is denoted by the superscript $H$. See [6] for detailed derivations. Dispersion curves can be calculated by solving the eigenvalue problem in eq.(14) for each wavenumber. Bandgaps will be evident in both the finite system’s receptance and infinite system’s dispersion curves. The discrete natural frequencies of a finite system are points on the dispersion curves of the infinite system. Thus a band gap will appear as a frequency zone of zero transmission or minima in the receptance function of the finite system [7].

**RESULTS AND DISCUSSION**

Consider first the dispersion curves shown in Fig. 3 corresponding, respectively, to the infinite versions of the systems shown in Fig. 1. The dashed lines in Fig. 3 correspond to the natural frequencies of the unitcell of the system with guided end boundary conditions or “free” natural frequencies while the solid horizontal lines correspond to “locked” natural frequencies. It can be verified that the locked and free natural frequencies of the unitcells form the bounds for the dispersion curves. Further insights can be obtained by considering each case of Fig. 1 individually, starting from an infinite beam without any periodicity.

An infinite beam without periodicity does not exhibit any characteristic stop or passbands. This can be explained based on the natural frequencies of a ‘unitcell’. First, in the simply supported condition, the vertical displacements of the unitcell edges are locked. It is well known [17] that the natural frequencies of a simply supported uniform beam and guided uniform beam are identical, except for a zero frequency rigid-body mode present in the guided beam. Thus, no bandgaps should exist in an infinite beam as expected. Consider next a beam with periodic masses as shown in Fig. 1(a). The natural frequencies of the unitcell — beam with a mass placed at its mid span under pinned and guided boundary condition — set the frequency bounds for passbands. Recall that the principal flexural modes of a simply supported beam are sine curves with a node in the middle for odd numbered modes, while those of guided beams are cosine curves [16, 17] with a node in the middle for even numbered modes. Hence, the odd modes of the guided beam and the even modes of the simply supported beam are unchanged by the addition of central mass in the unitcell. However, the separation of the natural frequencies between the even modes of the guided beam and the odd modes of the simply supported beam unitcell leads to the emergence of bandgaps. It follows then that the bandgap width is equal to the difference between the natural frequencies of the unitcell with pinned and guided boundary conditions. A similar analysis explains the appearance of a band-gap due to periodic local resonators system sketched in Fig. 1(b). The elastic coupling between the resonator mass and the rigid-body mode of the guided beam leads to the non-trivial result of the appearance of two frequencies: a zero frequency rigid mode and a finite frequency mode which sets the lower boundary for the sub-Bragg bandgap as shown in Fig. 3. The
Figure 3: Dispersion curves for the beam with periodic masses and/or resonators. Solid and dashed lines correspond to the natural frequencies of the unitcell under simply supported and guided boundary conditions, respectively.

Suppose a bandgap is desired around a frequency that lies below the Bragg frequencies, for example, a target frequency of 70 Hz. This clearly lies within the first passband of the beam with periodic masses in Fig. 3. The desired sub-Bragg bandgap can be created by replacing the masses with periodic spring-mass systems in each unitcell as shown in Fig. 2, each tuned to a resonant frequency of 70 Hz. Interference between a propagating wave in the beam (background medium) with period local resonators (spring-mass system) leads to the creation of sub-Bragg or local resonant bandgaps as shown in Fig. 3 for the case of beam with periodic resonators. It can also be noted that the widths of the Bragg bandgaps lying above the local resonance band gap diminish due to the closing-in of the locked and free natural frequencies of the unitcell as discussed earlier. Introducing a two-fold periodicity via periodic masses and resonators will help increase the width of Bragg bandgaps as shown in Fig. 3 for the case of beam with periodic masses and resonators.

Next, consider the finite periodic systems in Fig. 1. Recall that the normal modes of vibration of the finite periodic system are discrete points on the dispersion curves of the corresponding infinite system [7]. The bandgaps will now appear as minima in the finite system's receptance function, also known as frequency response function (FRF). A finite element model of the finite periodic beam was constructed and the transfer receptance between an input applied at one end and the response measured at the other end of the periodic beam is computed. The resulting FRFs are shown in Fig. 4 for each case. Notice that a bandgap of 70 Hz is formed at a frequency close to the target frequency for the case of a beam with periodic resonators. The
shift in the anti-resonance away from the target frequency is attributed to the Fano interferences between the scattering of the resonant modes and the propagating waves as noted in [18, 19]. It is worth pointing that a single tuned mass vibration absorber [17] also introduces a minimum in the FRF, but the structure of the FRF is less complex due to the absence of Fano interference due to resonant scattering caused by periodic resonators. It may be noticed again that the price paid to introduce sub-Bragg bandgap is the narrowing of the bandgaps corresponding to the Bragg frequencies. The two-fold periodic arrangement shown in 2 and the corresponding FRF is shown in Fig. 4. The bandgaps corresponding to the Bragg frequencies are now opened up due to the dominance of the periodicity of the masses over the periodicity of the resonators.

Finally, the width of the first sub-Bragg bandgap induced by the local resonator, computed from eq.(13),
Figure 5: Dependence of local resonance induced bandgap width on mass and stiffness ratios.

is a function of two non-dimensional parameters: mass ratio \( m_r = m_a/m_b \) and the stiffness ratio \( k_r = k_a/k_b \). Where \( m_b = \rho A l \) and \( k_b = \pi^4 EI/l^2 \) are respectively the modal mass and stiffness of the pinned-pinned beam for the first mode. It can be concluded from Fig. 5 that in order to achieve the largest possible bandgap, the stiffness and mass ratios \( k_r \) and \( m_r \) should be as high as possible.

**Conclusions**

This work has examined bandgap mechanisms in periodic systems with and without locally resonant structures. Receptance coupling technique is shown to provide a concise approach to calculate the width and location of Bragg and sub-Bragg bandgaps induced by a local resonator. This technique has revealed the surprising importance of rigid body (zero frequency) modes. It has been demonstrated that a local resonator narrows the width of high frequency bandgaps that would otherwise be present if the masses were not resiliently supported. This was explained on the basis of the modeshapes of the unitcell under pinned and guided boundary conditions. It is concluded that for a fixed target frequency (local resonance) of the sub Bragg bandgap, the width can be increased by proportionally increasing the mass (and the stiffness) of the resonator. This implies that stronger coupling and heavier resonator masses yield wider sub Bragg bandgaps.

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**References**


