4pPAa3. Perturbation methods for the spectral analysis of a weakly nonlinear acoustic field generated by a transient insonation

Hassina Khelladi* and Fahim Rahmi

*Corresponding author's address: Université des Sciences et de la Technologie Houari Boumediene, Alger, 16111, Bab Ezzouar, Algeria, hassinakhelladi@yahoo.fr

In this study an infinite plane piston is considered which oscillates with finite amplitude in unbounded homogeneous fluid. To illustrate the shape of the weakly nonlinear acoustic field generated by a transient insonation, the function defined by Funch/Müller representing a damped sinusoid is used to simulate the temporal waveform of the piston vibration. The acoustic transient wave generates harmonic components as result of nonlinearities in the material properties of the fluid and in the convective term of the propagation equation. The mathematical approach is based upon the generalized Burgers equation which is a good approximation of the exact equation for the nonlinear propagation when diffraction effects are assumed negligible. The pressure amplitude of the fundamental is considered large enough to produce the second harmonic wave. Under the quasi-linear approximation, an analytical description of the fundamental and the second harmonic waves is elaborated. To simulate the spectrum of the weakly nonlinear acoustic field, the pressure field is written in a perturbation series where the first term is the linear acoustic field that results from an infinitesimal oscillation of the piston and the second term contains the first nonlinear contribution to the acoustic field due to the finite amplitude effects.

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INTRODUCTION

Acoustic pulses with high intensity are used in many practical applications such as Tissue Harmonic Imaging (THI) or second harmonic imaging which offers several advantages over conventional pulse-echo imaging. Both harmonic contrast and lateral resolution are improved in harmonic mode. Tissue Harmonic Imaging also provides a better signal to noise ratio which leads to better image quality in many applications. The major benefit of Tissue Harmonic Imaging is artifact reduction resulting in less noisy images, making cysts appear clearer and improving visualization of pathologic conditions and normal structures. Indeed, Tissue Harmonic Imaging is widely used for detecting subtle lesions (e.g., thyroid and breast) and visualizing technically-challenging patients with high body mass index. To create images exclusively from the second harmonic wave, currently the pulse inversion technique is used. So, to optimize the image quality, detailed knowledge of the second harmonic wave of the acoustic field as function of the pulse shape is needed.

In general, high intensity sound reveals finite amplitude effects, which linear acoustical theory cannot predict. Thus, the investigation of finite amplitude wave propagation in thermoviscous fluid requires appropriate consideration of the combined effects due to absorption, nonlinearity and diffraction. In the present study, the mathematical approach used is based upon the generalized Burgers’ equation which is a good approximation of the exact equation for the nonlinear propagation when diffraction effects are assumed negligible. The pressure amplitude of the fundamental is considered large enough to produce the second harmonic wave. Under the quasi-linear approximation, an analytical description of the fundamental and the second harmonic waves is elaborated. For that purpose, an infinite plane piston is considered that oscillates with finite amplitude in unbounded homogeneous fluid. To illustrate the shape of the weakly nonlinear acoustic field generated by a transient insonation, the function defined by Funch/Müller representing a damped sinusoid is used to simulate the temporal waveform of the piston vibration. The acoustic transient wave generates harmonic components as result of nonlinearities in the material properties of the fluid and in the convective term of the propagation equation. To simulate the spectrum of the weakly nonlinear acoustic field, the pressure field is written in a perturbation series where the first term is the linear acoustic field that results from an infinitesimal oscillation of the piston and the second term contains the first nonlinear contribution to the acoustic field due to the finite amplitude effects. This investigation is based on Krassilnikov and coworkers experimental results [1]. These experimental data concern glycerol which corresponds to a strongly dissipative fluid with some similarities to soft tissues [2]. The utility of the method resides in the ease with which it can be implemented on a digital computer.

THEORY

The propagation of finite amplitude plane progressive waves in a homogeneous and dissipative liquid is governed by the Burgers’ equation. Here, it is assumed that the ultrasonic wave propagates in the positive $z$ direction, and the differential change of the acoustic pressure $p$ with respect to $z$ can be written as follows [3]:

$$
\left(-\frac{2}{c_0}\frac{\partial}{\partial t} + \frac{D}{c_0^2}\frac{\partial^2}{\partial t^2}\right)p = -\frac{\beta}{\rho_0 c_0} \left(\frac{\partial^2}{\partial t^2} p^2 \right)
$$

(1)

$$
D = \frac{1}{\rho_0} \left[\frac{4}{3}(\mu + \xi) + \kappa \left(\frac{1}{c_v} - \frac{1}{c_p}\right)\right]
$$

is the diffusivity of the sound for a thermoviscous fluid. This parameter is a function of the fluid shear viscosity $\mu$, the fluid bulk viscosity $\xi$, the thermal conductivity $\kappa$, the specific heat at constant volume $c_v$ and the specific heat at constant pressure $c_p$. The acoustic nonlinearity parameter $\beta = 1 + B/2A$ is function of the nonlinearity parameter of the medium $B/A$, which represents the ratio of quadratic to linear terms in the isentropic pressure-density relation [4-7]. $\tau = t - z/c_0$ is the retarded time, $c_0$ is the infinitesimal sound velocity and $\rho_0$ is the undisturbed density of the liquid.
The first term on the left hand side of equation (1) is the linear wave propagation, while the second term represents the loss due to viscosity and heat conduction or any other agencies of dissipation. The term on the right hand side of equation (1) is the nonlinear term that accounts for quadratic nonlinearity, producing cumulative effects in progressive plane wave propagation.

As mentioned above, we consider a plane piston that oscillates at a frequency \( \omega \) with small but finite amplitude in a fluid that is unbounded and at rest at infinity. The piston creates an acoustic wave in the fluid that propagates in the axial direction. The acoustic wave generates harmonic components as result of nonlinearities in the material properties of the fluid and in the convective term of the propagation equation. Thus, the amplitude of the acoustic wave decreases as it propagates in the fluid due to the dissipative effects. The acoustic wave also generates harmonic waves as a result of nonlinearities in the material properties of the fluid and in the convective term of the equations of motion. At first, the amplitudes of the harmonic waves increase in magnitude and then decrease as they propagate in the fluid. To describe these waves, the acoustic pressure is written in a perturbation series [3, 8-9]:

\[
p = M \ p_1 + M^2 \ p_2 + \theta (M^3)
\]  
(2)

where \( M = \frac{U_0}{c_0} \) is the acoustic Mach number, which represents the ratio of the particular velocity amplitude to the considered fluid sound velocity. The Mach number quantifies the importance of the nonlinear phenomena in lossless fluid.

The first term on the right hand side of equation (2) is the linear acoustic pressure field that results from an infinitesimal oscillation of the piston. The second term contains the first nonlinear contribution to the acoustic pressure field due to finite amplitude effects.

The following set of equations is obtained by substituting equation (2) into equation (1) and equating coefficients of like power of M:

\[
\begin{aligned}
\left( \frac{\partial}{\partial z} - \frac{D}{2c_0^3} \frac{\partial^2}{\partial \tau^2} \right) p_1 &= 0 \\
\left( \frac{\partial}{\partial z} - \frac{D}{2c_0^3} \frac{\partial^2}{\partial \tau^2} \right) p_2 &= \frac{\beta}{2p_0 c_0^3} \left( \frac{\partial p_1^2}{\partial \tau} \right)
\end{aligned}
\]  
(3)

Referring to equation (3), the linear part \( p_1(z, \tau) \) of the solution does not seem to be influenced by the nonlinear effects. Indeed, in the case of the quasi-linear approximation the extracted energy from \( p_1(z, \tau) \) due to the nonlinear phenomena is considered weak compared to the dissipated energy in the medium.

As for the quasi-linear part \( p_2(z, \tau) \) of the solution, it is determined by equation (3). It is shown that the term on the right hand side of equation (3) is a quadratic source term resulting from the propagation of the linear acoustic pressure field \( p_1(z, \tau) \). Thus, in the case of the quasi-linear approximation, \( p_2(z, \tau) \) propagates linearly while being maintained by the nonlinear sources present in the propagation medium.

The use of the perturbation method and the quasi-linear approximation make possible to solve the nonlinear nature of the problem by uncoupling it.

In order to illustrate the shape of the pressure field radiated by a plane piston that oscillates at a frequency \( \omega \) with small but finite amplitude in a fluid that is unbounded and at rest at infinity, the function defined by Funch/Muller representing a damped sinusoid containing N cycles is used to represent the temporal waveform of the piston vibration \( U(t) \) [10-11].
where \( N \) is the number of cycles and \( \tau \) is the pulse duration \( 2\pi N/\omega \).

An analysis of the pressure field is undertaken in the case of a strongly dissipative fluid by using glycerol experimental data given by Krassilnikov and coworkers [1].

During simulations all precautions related to the signal processing (sampling, theorem of Shannon and causality principle...) are rigorously respected. All discussions, related to the analysis of the effect of some parameters on the spectrum of the weakly nonlinear acoustic field, are reported only on the nonlinear part of the ultrasonic wave propagation in the analyzed fluid.

### NUMERICAL EXPERIMENTS AND DISCUSSIONS

Krassilnikov’s experimental data for glycerol are used in order to simulate the spectrum of the weakly nonlinear acoustic field by using the analytical solution of equation (3). Table 1 lists material properties.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Temperature (°C)</th>
<th>Density (kg/m³)</th>
<th>Sound velocity (m/s)</th>
<th>Acoustic nonlinearity parameter</th>
</tr>
</thead>
<tbody>
<tr>
<td>Glycerol</td>
<td>20</td>
<td>1260</td>
<td>1980</td>
<td>5.4</td>
</tr>
</tbody>
</table>

According to Krassilnikov and coworkers experimental study, the absorption coefficient is a quadratic function of frequency. The absorption coefficient is that obtained from an infinitesimal acoustic excitation, even though the acoustic intensity increases, in the case of glycerol \( \alpha_0 = 2610^{-13} \text{ Np.m}^{-1}.\text{Hz}^{-2} \) (Table 2).

To simulate the spectrum of the weakly nonlinear acoustic field, the pressure field is written in a perturbation series where the first term is the linear acoustic field that results from an infinitesimal oscillation of the piston and the second term contains the first nonlinear contribution to the acoustic field due to the finite amplitude effects (equation (2)). The considered temporal waveform of the piston vibration is a damped sinusoid function defined by Funch/Muller (equation (4)). The acoustic transient wave generates harmonic components as result of nonlinearities in the material properties of the fluid and in the convective term of the propagation equation.

By using the perturbation methods and the use of FFT which alternates between time and frequency domains, the spectrum of the weakly nonlinear acoustic field generated by a transient insonation has been simulated.

<table>
<thead>
<tr>
<th>Temperature (°C)</th>
<th>Intensity (W/cm²)</th>
<th>Shock length (m)</th>
<th>Goldberg Number ( \Gamma )</th>
</tr>
</thead>
<tbody>
<tr>
<td>22</td>
<td>0.2</td>
<td>1.443</td>
<td>0.07</td>
</tr>
<tr>
<td>21</td>
<td>2.5</td>
<td>0.408</td>
<td>0.24</td>
</tr>
<tr>
<td>19</td>
<td>4.5</td>
<td>0.304</td>
<td>0.32</td>
</tr>
</tbody>
</table>

\[ U(t) = \begin{cases} U_0 \left( \sin(\omega t) - \left( \frac{N + 1}{N} \right) \sin \left( \frac{N + 1}{N} \omega t \right) \right) & 0 \leq t < \tau \\ 0 & \text{otherwise} \end{cases} \]
The effect of the oscillation frequency of an infinite plane piston on the spectrum of the weakly nonlinear acoustic field is illustrated in figure 1, in which the increase in frequency induces a widening of the bandwidth of the spectral component of the fundamental pressure field and that of the generated second harmonic. Referring to figure 1, a decrease in magnitude of the spectral components is observed by increasing the oscillation frequency. At \( f = 2\text{MHz} \) the spectral component magnitude of the second harmonic is practically negligible, thus the nonlinear effect becomes masked by the effect of absorption. In addition, it should be noted that the decrease in frequency results in an appreciable deformation of the signal which is in conformity with the nonlinear acoustic theory. Indeed, as the number of Goldberg is conversely proportional to the oscillation frequency, to decrease this latter, reinforce the nonlinear effects by increasing the Goldberg number.

**FIGURE 2.** Effect of the location \( z \) on the spectrum of the weakly nonlinear acoustic field: \( N = 10, f = 1\text{MHz}, U_0 = 30 \text{ m/s} \).
Figure 2 represents the effect of the location $z$ on the spectrum of the weakly nonlinear acoustic field. Referring to figure 2, a decrease in magnitude of the spectral components is observed by increasing $z$. It is evident that by rising $z$, the deformation rate of the acoustic wave becomes insignificant due to the absorption phenomenon which masks practically the nonlinear effects.

A simulation of the effect of the vibration amplitude of an infinite plane piston on the spectrum of the weakly nonlinear acoustic field is represented in figure 3. It is obvious that an increase of the vibration amplitude of the piston induces an increase of the magnitude of the fundamental spectral component involving consequently a reinforcement of the spectral component magnitude of the generated second harmonic. Thus, the generated harmonic can only follow the evolution of the fundamental which gives it birth. In addition, it should be noted that the increase in the vibration amplitude results in an appreciable deformation of the signal which is in conformity with the nonlinear acoustic theory. Indeed, as the number of Goldberg is proportional to the vibration amplitude, to increase this latter, reinforce the nonlinear effects by increasing the Goldberg number.

Figure 4. Effect of the number of cycles on the spectrum of the weakly nonlinear acoustic field: $z = 0.1$ m, $f = 1$ MHz, $U_0 = 30$ m/s.
Figure 4 illustrates the effect of the duration of the piston oscillation on the spectrum of the weakly nonlinear acoustic field. For low values of number of cycles which represent a transient source condition (broad band signal), the component of the fundamental pressure field and that of the generated second harmonic field overlap, generating a modified broad band spectrum. When the duration of oscillation becomes significant, the two spectral components become distinct (Figure 4). This configuration is equivalent to a harmonic source condition with narrow band. Referring to Figure 4, it should be noted that the amplitude of the signal grows when the number of cycles increases. This can be explained by the analytical formulation of the function defined by Funch / Muller. Indeed, it is noticeable that when the number of cycles increases, the amplitude undergoes a light increase.

CONCLUSION

The use of the perturbation method and the quasi-linear approximation make possible to solve the nonlinear nature of the problem by uncoupling it. Indeed, by using the perturbation methods and the use of FFT which alternates between time and frequency domains, the spectrum of the weakly nonlinear acoustic field generated by a transient insonation has been simulated. 2D spectrum of the weakly nonlinear acoustic field versus the cycle number of the dumped sinusoid illustrates an overlap between the fundamental bandwidth and the second harmonic frequency range when lower cycle numbers are considered. In these cases, the second harmonic frequency band is affected with some frequency components of the fundamental. Although, broad bandwidth implies a short pulse length and improved axial resolution, sometimes a compromise must be made in the choice of the cycle number despite of the pulse inversion technique implemented in the harmonic imaging system.

The proposed approach can be also extended by taking into account the diffraction effects in the simulation of the weakly nonlinear pulsed field.

REFERENCES