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1pPPb20. Sensory consonance of two simultaneous sine-tones
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In Chapter 4 of my book 'Musical Consonance and Cochlear Mechanics' (vdf, Zurich, 2012), four psychoacoustic experiments on the sensory consonance of two simultaneous sine-tones are described. In each of those experiments, the deeper-tone frequency \( f_d \) was kept fixed, at \( f_d = 132, 264, 528, \) or 1056 Hz. The resulting consonance curves (sensory consonance versus higher-tone frequency \( f_h \)) exhibit consonance minima at beat-rates \( b_{\text{m.d.}} = f_h - f_d \) (where ‘m.d.’ = ‘most dissonant’) ranging from 13 Hz (at \( f_d = 132 \) Hz) to 39 Hz (at \( f_d = 1056 \) Hz). In Section 15.1 of the above-mentioned book, these most dissonant beat rates are shown to agree well with the following empirical law: \( b_{\text{m.d.}} = (1.07s^{-0.5}) \times \sqrt{f_{\text{avg}}}, \) where \( f_{\text{avg}} = f_d + b_{\text{m.d.}}/2. \) The just described empirical law is unsatisfactory because in the underlying experiments the deeper-tone frequency \( f_d \) [rather than the average frequency \( (f_d + f_h)/2 \)] was kept constant. It is found that the data agree equally well with the following modified empirical law: \( b_{\text{m.d.}} = (1.1s^{-0.5}) \times \sqrt{f_d}. \) This modification does not affect the validity of the complex-tone consonance theories described in Chapters 15 and 16 of the mentioned book.

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1. INTRODUCTION

The sound-pressure-versus-time function of a stationary sine-tone of frequency $f$ at a given place in air is defined by the following equation:

$$p(t) = p_0 \cdot \sin(\omega t + \varphi);$$  \hspace{1cm} (1)

here $\omega = 2\pi f$ is the angular frequency; $p_0$ and $\varphi$ are constants. The sensory consonance (i.e., the pleasantness or beauty) of two simultaneous sine-tones of frequencies $f_d, f_h$ differs strongly from the sensory consonance of two simultaneous harmonic complex tones of the same frequencies $f_d, f_h$ if the complex tones have strong overtones. Nevertheless, the sine-tone consonances strongly influence the complex-tone consonances; according to the Helmholtz consonance theory [1], a pair of simultaneous complex tones is dissonant if in the combined partial-tone spectrum there are pairs of strong partial tones having frequencies such that the rate of the generated beats is in a certain range. That range depends on the frequency $f_d'$ of the deeper of the two considered partial tones; e.g., the most dissonant beat rate (i.e., the most dissonant difference between the two partial-tone frequencies) for partial-tone frequencies near 100 Hz is about 10 Hz (e.g., $f_d' = 100$ Hz, $f_h = 110$ Hz), and for partial-tone frequencies near 400 Hz it is about 20 Hz (e.g., $f_d' = 400$ Hz, $f_h = 420$ Hz). The topic of the present note is the dependence of the sensory consonance on the frequency $f_h'$ of the higher sine-tone for a fixed deeper-sine-tone frequency $f_d$ between 100 Hz and 1 kHz.

In Chapter 4 of my recently published book “Musical Consonance and Cochlear Mechanics” [2], four corresponding psycho-acoustic “one-man” experiments are described. The deeper-sine-tone frequency $f_d$ in these experiments amounted to 132, 264, 528, or 1056 Hz. The results of the four experiments are displayed in Figs. 1-4 below. The abscissa in these diagrams gives the interval $x$ between $f_d'$ and $f_h'$ in cents; i.e.,

$$x = (1200 \text{ cents}) \cdot \log(f_h'/f_d') / \log(2);$$ \hspace{1cm} (2)

e.g., if $f_h'/f_d' = 2$, then $x = 1200$ cents; if $f_h'/f_d' = 1.1$, then $x = 165$ cents. The consonance ratings follow the Swiss school-grade system; rating 6 means “very consonant”; and rating 1 means “very dissonant”.

![Figure 1: Consonance ratings of two simultaneous sine tones; deeper-tone frequency $f_d = 132$ Hz. Solid line: sound-pressure level of sine tones 50 dB; dashed line: 70 dB.](image-url)
FIGURE 2. Consonance ratings of two simultaneous sine tones; deeper-tone frequency $f_d = 264$ Hz. Solid line: sound-pressure level of sine tones 50 dB; dashed line: 70 dB.

FIGURE 3. Consonance ratings of two simultaneous sine tones; deeper-tone frequency $f_d = 528$ Hz. Solid line: sound-pressure level of sine tones 50 dB; dashed line: 70 dB.

FIGURE 4. Consonance ratings of two simultaneous sine tones; deeper-tone frequency $f_d = 1056$ Hz. Solid line: sound-pressure level of sine tones 50 dB; dashed line: 70 dB.
2. THE MOST DISSONANT BEAT-RATES

In the remainder of this note, the most dissonant beat rate $b_{m.d}$ is defined to be given by the center $x_{m.d}$ of the interval range in which the rating 1 (very dissonant) was obtained. The most dissonant beat-rates derived from Figs. 1-4 are presented in Tables 1 and 2 below.

**TABLE 1.** Most dissonant beat rates derived from Figs. 1-4; sound-pressure level of sine-tones: 50 dB.

<table>
<thead>
<tr>
<th>$f_d$ [Hz]</th>
<th>Fig.</th>
<th>$x_{m.d}$ [c.]</th>
<th>$\Delta x_{m.d}$ [c.]</th>
<th>$R_{m.d}$</th>
<th>$f_h$ [Hz]</th>
<th>$b_{m.d}$ [Hz]</th>
<th>$\Delta b_{m.d}$ [Hz]</th>
</tr>
</thead>
<tbody>
<tr>
<td>132</td>
<td>1</td>
<td>162.5</td>
<td>12.5</td>
<td>1.0984</td>
<td>145.0</td>
<td>13.0</td>
<td>1.0</td>
</tr>
<tr>
<td>264</td>
<td>2</td>
<td>112.5</td>
<td>12.5</td>
<td>1.0671</td>
<td>281.7</td>
<td>17.7</td>
<td>2.0</td>
</tr>
<tr>
<td>528</td>
<td>3</td>
<td>75.0</td>
<td>12.5</td>
<td>1.0443</td>
<td>551.4</td>
<td>23.4</td>
<td>4.0</td>
</tr>
<tr>
<td>1056</td>
<td>4</td>
<td>62.5</td>
<td>12.5</td>
<td>1.0368</td>
<td>1094.8</td>
<td>38.8</td>
<td>7.9</td>
</tr>
</tbody>
</table>

**TABLE 2.** Most dissonant beat rates derived from Figs. 1-4; sound-pressure level of sine-tones: 70 dB.

<table>
<thead>
<tr>
<th>$f_d$ [Hz]</th>
<th>Fig.</th>
<th>$x_{m.d}$ [c.]</th>
<th>$\Delta x_{m.d}$ [c.]</th>
<th>$R_{m.d}$</th>
<th>$f_h$ [Hz]</th>
<th>$b_{m.d}$ [Hz]</th>
<th>$\Delta b_{m.d}$ [Hz]</th>
</tr>
</thead>
<tbody>
<tr>
<td>132</td>
<td>1</td>
<td>187.5</td>
<td>12.5</td>
<td>1.1444</td>
<td>147.1</td>
<td>15.1</td>
<td>1.0</td>
</tr>
<tr>
<td>264</td>
<td>2</td>
<td>100.0</td>
<td>12.5</td>
<td>1.0595</td>
<td>279.7</td>
<td>15.7</td>
<td>2.0</td>
</tr>
<tr>
<td>528</td>
<td>3</td>
<td>62.5</td>
<td>12.5</td>
<td>1.0368</td>
<td>547.4</td>
<td>19.4</td>
<td>4.0</td>
</tr>
<tr>
<td>1056</td>
<td>4</td>
<td>62.5</td>
<td>12.5</td>
<td>1.0368</td>
<td>1094.8</td>
<td>38.8</td>
<td>7.9</td>
</tr>
</tbody>
</table>

The quantity $R_{m.d}$ in the fifth column of Tables 1 and 2 is the frequency ratio $(f_h / f_d)$ corresponding to the interval size $x_{m.d}$ given in the third column; i.e.,

$$R_{m.d} = 2^{x_{m.d}/1200}.$$  

(3)

The uncertainty $\Delta x_{m.d}$ (fourth column) was set to one half of the step size $(\Delta x = 25$ cents) used in the experiments that yielded Figs. 1-4. The uncertainty $\Delta b_{m.d}$ is related to the uncertainty $\Delta x_{m.d}$ as follows:

$$\Delta b_{m.d} = (db/dx) \cdot \Delta x_{m.d};$$  

(4)

the derivative in Eq. (4) must be evaluated at $x = x_{m.d}$ for the following function:

$$b(x) = f_d \cdot (2^{x/1200} - 1);$$  

(5)

Eqs. (3), (4), and (5) yield the following uncertainty $\Delta b_{m.d}$:

$$\Delta b_{m.d} = \ln(2) \cdot f_d \cdot \Delta x_{m.d} / 1200.$$  

(6)

The uncertainties $\Delta b_{m.d}$ evaluated according to Eq. (6) are listed in column 8 of Tables 1 and 2.

3. EMPIRICAL LAWS FOR THE MOST DISSONANT BEAT RATE


In Eq. (15.1) of the mentioned book [2], the following law for the most dissonant beat rate was presented:

$$b_{m.d} = k_d \cdot \sqrt{f_{avg}}.$$  

(7)

Here the quantity $f_{avg}$ is the average frequency, i.e. $f_{avg} = (f_d + f_h)/2$. The constant $k_d$ was determined to be $(1.067\pm0.053)s^{1/2}$ for 50-dB sine-tones, and $(1.078\pm0.100)s^{1/2}$ for 70-dB sine-tones. The comments of a reader of the book [2] made me realize, however, that Eq. (7) is unsatisfactory: in the experiments that yielded Figs. 1-4 of the present note, the deeper-tone frequency $f_d$ (rather than the average frequency $f_{avg}$) was kept constant. In the following section, an empirical law involving $f_d$ instead of $f_{avg}$ is presented.
3.2. A Modified Empirical Law

The new law is similar to Eq. (7):

\[ b_{th}(i) = k \cdot \sqrt{f_a(i)} \]  

(8)

Here, the subscript “th” stands for “theoretical”, and the integer \( i \) (1, 2, 3, or 4) designates the running number of the line in Table 1 or 2; e.g. \( f_d(1) = 132 \text{ Hz} \). In Fig. 5, the following function is displayed:

\[ \chi^2(k) = \sum_{i=1}^{4} \frac{[b_{th}(i) - b_{m,d}(i)]^2}{[\Delta b_{m,d}(i)]^2}. \]  

(9)

The quantity \( b_{th}(i) \) in Eq. (9) is calculated, for any given value of the constant \( k \), by means of Eq. (8); the experimental quantities \( b_{m,d}(i) \) and \( \Delta b_{m,d}(i) \) in Eq. (9) are taken from Table 1 or 2.

**FIGURE 5.** The functions \( \chi^2(k) \) defined by Eq. (9); solid curve: sine-tone level 50 dB, Table 1; dashed curve: sine-tone level 70 dB, Table 2.

The best value of the constant \( k \) is given by the point of minimal \( \chi^2(k) \) in Fig. 5, and the uncertainty \( \Delta k \) is given by the points where \( \chi^2(k) = \chi^2(\text{minimal}) + 1 \); resulting values of the constant \( k \):

\[ k \text{ (50dB)} = (1.11 \pm 0.07)s^{-0.5}; \quad k \text{ (70dB)} = (1.14 \pm 0.07)s^{-0.5}. \]  

(10)

The minimal \( \chi^2 \)-values and the corresponding \( \chi^2 \)-probabilities \( P \) (for 4-1=3 degrees of freedom) are:

\[
\begin{align*}
50 \text{ dB: } & \chi^2 = 0.48; \quad P = 0.923; \quad 70 \text{ dB: } \chi^2 = 8.50; \quad P = 0.037.
\end{align*}
\]  

(11)

The high probability (0.923) at 50 dB implies that Eq. (8) fits the 50-dB data very well; at 70 dB (\( P = 0.037 \)) the fit is less good. For comparison, the corresponding \( \chi^2 \)-probabilities in the case of the older empirical law defined by Eq. (7) were \( P = 0.934 \) (at 50 dB) and \( P = 0.050 \) (at 70 dB), not differing strongly from the new probabilities given in Eq. (11).

The experimental functions \( b_{m,d}(f_d) \) according to Tables 1 and 2 and the theoretical functions \( b_{th}(f_d) \) according to Eqs. (8) and (10) are shown in Figs. 6 and 7 below.
4. CONCLUSIONS

In Section 15.1 of “Musical Consonance and Cochlear Mechanics” [2], it was found that the most dissonant beat-rate of two simultaneous sine-tones of 50 or 70 dB can be approximated by the following empirical law:

\[ b_{m.d.} \approx (1.07 s^{-1}) \cdot \sqrt{f_{\text{avg}}} \]  

(12)

As discussed in Section 3.1 above, it is preferable to replace Eq. (12) by a similar law involving, instead of \( f_{\text{avg}} \), the frequency \( f_d \) of the deeper of the two simultaneous sine-tones. In the present study the following new empirical law for the most dissonant beat-rate has been found [see Eqs. (8) and (10)]:

\[ b_{m.d.} \approx (1.1 s^{-1}) \cdot \sqrt{f_d} \]  

(13)

FIGURE 6. Solid curve: the “theoretical” function \( b_{th}(f_d) \) according to Eqs. (8) and (10), at a sine-tone level of 50 dB; points with error bars: experimental values of \( b_{m.d.}(f_d) \) according to columns 7 and 8 of Table 1.

FIGURE 7. Dashed curve: the “theoretical” function \( b_{th}(f_d) \) according to Eqs. (8) and (10), at a sine-tone level of 70 dB; points with error bars: experimental values of \( b_{m.d.}(f_d) \) according to columns 7 and 8 of Table 2.
Eq. (13) fits the 50-dB data (Table 1, Fig. 6) very well, whereas in the case of the 70-dB data (Table 2, Fig. 7) the fit is mediocre.

The replacement of Eq. (12) by Eq. (13) does not affect the validity of the complex-tone consonance theories described in Chapters 15 and 16 of “Musical Consonance and Cochlear Mechanics” [2]. The most dissonant beat rates according to Helmholtz are lower than those defined in Eq. (13) by a factor of ~0.7; in Section 15.1 of [2], however, it is shown that the just mentioned deviation tends to be compensated by the fact that in the Helmholtz theory, as compared with modern experiments, the roughness contributed by pairs of high partial tones is underestimated. For example, the well-known dissonance curve in Fig. 60a of [1] (reproduced in Fig. 8 below) contains a total-dissonance peak slightly to the left of the higher complex tone g'. An increase in the assumed most dissonant partial-tone beat rates shifts that peak to the left; if the height of the partial-dissonance peaks 4:6 and 6:9 is increased, the mentioned total-dissonance peak is shifted back to the right.

**FIGURE 8.** Reproduction of Fig. 60a of [1]. Deeper bowed-string tone c' (256 Hz). Roughness (i.e., dissonance) is plotted versus logarithm of frequency of simultaneous higher bowed-string tone; for example, $f(g') = 384$ Hz; $f(c'') = 512$ Hz. The partial-roughness curve 2:3, e.g., displays the roughness due to beats generated by the third partial of the deeper complex tone and the second partial of the higher complex tone.

**REFERENCES**
