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1aSP2. Optimal beamformer designed for robustness against channel mismatch based on Monte Carlo Simulation

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Design of beamformers that withstand mismatch in channel characteristics (gain, phase and position) and pointing error (rotational) has been a key issue in array signal processing. These mismatch factors are random in nature and generally intractable by deterministic approaches. This paper examines these effects on beamformer performance from a statistical perspective. The aim of this work is twofold: analysis and synthesis. In the analysis phase, the mismatch factors of microphone characteristics are assumed to be random variables following either uniform or Gaussian distribution. Statistics including the mean, maximum, minimum and the maximum likelihood (ML) of performance measures (directivity index and white noise gain) are efficiently obtained via Monte-Carlo Simulation (MCS). This provides useful information for choosing performance measures in the next synthesis phase. Optimal parameters of superdirective array designed using least squares (LS) and convex optimization (CVX) are determined based on the preceding performance measures. Simulation results have shown that the proposed statistical approach with different performance measures provided various degrees of performance-robustness tradeoffs in the optimal beamformer design.

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Optimal Beamformers designed for Robustness against Channel Mismatch based on Monte Carlo Simulation

I. INTRODUCTION

One of the key issues in beamformer design is how to deal with system errors due to channel mismatch (gain, phase and position) and pointing error (rotational). Recent evidence suggests that beamformers such as Delay-and-Sum (DAS) algorithm, Practical Supergain Method, Interior Point Method, Super-Directive Array (SDA), the Minimum Variance Distortionless Response (MVDR) array, and the robust Capon beamformer are meant for sources in long distance and of large size. Cox et al. suggested an optimal approach in which they described an array design method in the light of constrained two-norm optimization. The performance achieved by their array is comparable with unconstrained supergain value and conventional beamforming. Lebret employed Interior Point Method in antenna array pattern design. In the SDA design, directivity and robustness are traded off by weighting the reciprocals of the Directivity Index (DI) and White Noise Gain (WNG) of the array. Stoica et al. suggested a Robust Capon Beamformer (RCB) to tackle the pointing errors. Array weights are obtained by minimizing the array output variance, subjected to 2-norm or affine constraints formulated as uncertainty ellipsoid. Their approach requires the knowledge of array covariance matrix and therefore is data-dependent. Numerical simulation confirms that the RCB is capable of achieving beamforming performance as well as robustness against uncertain steering angles. Yan et al. have recently developed a methodology for the time-domain implementation of broadband beamformer in terms of spherical harmonics. A multiple constraint problem was formulated to tradeoff among multiple performance indices such as DI, WNG, the sidelobe level and the mainlobe response variation.

The purpose of this paper is to design the optimal beamformer by taking into account the statistic properties of the system errors. This paper divided optimal beamformer design into analysis and synthesis phases. In the analysis phase, Monte-Carlo simulation (MCS) is based on random search that involves random numbers and probability density functions (pdf). The major advantage of MCS is to efficiently evaluate high-dimensional integrals in statistics, which is generally intractable with numerical grid-based integration methods. MC methods are a class of computational algorithms that rely on random sampling to compute their results. Because of their dependence on the repeated computation of random numbers, the MCS is most suited to calculation by drawing random samples subject to the prescribed probability functions. In our application, uniform and normal distributions are used to simulate system errors. In the synthesis phase, the least-squares (LS) and convex optimization (CVX) approaches can be adopted to find optimal solutions. Two performance measure of beamformer is DI and Composite Index (CI, which combined with DI and WNG). In mathematical optimization, the quadratic form often arises as the objective function, or the cost function, in LS problems. CVX provides a useful alternative to conventional methods such as LS optimization and linear programming. In this paper, we shall use CVX in designing data-independent beamformers. The proposed design is compared to the LS approach.

Four statistics including the mean, maximum, minimum and the ML of performance measures (DI and CI) are compared with uniform and normal random samplings. The LS and CVX optimizations are carried out to tradeoff array directivity and robustness against sensor mismatch and noise. To evaluate the different synthesis criteria (max mean DI, max min DI, max ML for LS and max min CI for CVX) of optimal beamforming, simulation and experiment are undertaken, with the aid of MCS for system errors.

II. STATISTICAL ANALYSIS OF BEAMFORMERS

A. Farfield array model

For the farfield array model, we assume that the sources are located far enough from the array that the wave fronts impinging on the array can be modeled as spherical waves radiated by a point source. Consider a uniform linear array comprised of \( M \) microphones distributed in a linear lattice with inter-element spacing \( d \) in the \( x \) axis. By assuming the time-harmonic dependence \( e^{j\omega t} \), the sound pressure field at the \( m \)th microphone can be written as

\[
P(x_m, \omega) = s(\omega)e^{j\omega x_m/c} + n(x_m, \omega), \quad m = 1, 2, \cdots, M
\]

where \( j = \sqrt{-1} \), \( x_m \) is the position vectors of the \( m \)th microphone, \( s(\omega) \) is the Fourier transform of the source signal, wave vector \( \mathbf{k} = -\hat{k} \mathbf{c} = -\omega / c \mathbf{c} \) is the wave vector with \( \mathbf{c} \) being the unit vector pointing from the
array reference to the source, \( \bullet \) denotes inner product, the constant \( c \) is the speed of sound, \( \omega \) is the angular frequency, and \( n(x_m, \omega) \) is the uncorrelated sensor noise injected to the \( m \)th microphone signal. A beamformer can be regarded as a linear combiner of weighted microphone signals to yield the array output signals:

\[
y(\omega) = w^H(\omega)a(\omega)s(\omega) + w^H(\omega)n(\omega),
\]

where \( a(\omega, \theta) = \begin{bmatrix} 1 \ e^{j\omega \cos \theta} \cdots \ e^{j(M-1)\omega \cos \theta} \end{bmatrix}^T \) is called the array manifold vector, superscript “\( T \)” denotes matrix transpose, \( \theta \) is the look direction, \( n(\omega) = [n(x_1, \omega) \cdots n(x_M, \omega)]^T \), and \( w(\omega) = [w_1(\omega) \cdots w_M(\omega)]^T \) denotes the array weight vector, and superscript “\( H \)” denotes matrix hermitian transpose. Therefore, the combined sound field and array system can be regarded as a single-input-single-output linear system with \( s(\omega) \) and \( y(\omega) \) as the input and the output, respectively. In this context, the directional response or the array pattern is governed by the array response function, or directivity factor, \( G(\omega) = w^H a(\omega) \) that involves the dynamics of the acoustic propagation process \( a(\omega) \) and the array filters \( w(\omega) \).

### B. Performance measures of beamformers

In the following, we introduce the performance measures of beamformers, accompanied with the optimization method for designing array beam pattern. The DI is defined as the ratio of on-axis intensity to that of the intensity produced by a point source with equal power:

\[
DI = 10 \log \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi \left| G(\omega, \theta, \phi) \right|^2 \sin \theta d\theta d\phi = 10 \log \frac{w^H A w}{w^H B w},
\]

where

\[
A = a(\omega, \theta_0, \phi_0) a^H(\omega, \theta_0, \phi_0)
\]

\[
B = \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi a(\omega, \theta, \phi) a^H(\omega, \theta, \phi) \sin \theta d\theta d\phi
\]

\[
= \frac{1}{2} \int_0^\pi a(\omega, \theta) a^H(\omega, \theta) \sin \theta d\theta \quad \text{(if axi-symmetrical)}
\]

\( \theta \) and \( \phi \) are the angles in spherical coordinates, \( \theta_0 \) and \( \phi_0 \) are the angles of the acoustical axis. Another important performance measure is WNG that is the array gain due to spatially uncorrelated noise:

\[
WNG = 10 \log \frac{w^H A w}{w^H w},
\]

which can be regarded as a measure for robustness against sensor’s self noise. The third one performance measure is CI which combined with above DI and WNG for CVX approach:

\[
CI = DI + \alpha WNG,
\]

where \( \alpha \) is the normalization number. The parameter \( \alpha = 1/7 \) in the simulation.

### C. Monte-Carlo simulation of performance measures

The MCS is a simulation technique relying on drawing random samples subject to a prescribed distribution. More generally, MCS is useful for modeling and evaluating a deterministic model with uncertain parameters as inputs. The purpose of this section is how to approximate the desired pdf to simulate the system errors (microphone mismatches and pointing error).

The system errors of the \( m \)th microphone are given by:

\[
A_m(\omega, \theta) = a_m(\omega, \theta) e^{j\omega_m(\theta)} e^{j\omega_m(\theta)} e^{j\sigma_m(\theta)} e^{j\sigma_m(\theta)}
\]

where \( a_m(\omega, \theta) \), \( \psi_m(\omega, \theta) \) and \( \tau_m \) are gain, phase and position errors of microphone characteristics. \( d_m \cos \alpha_m \) is the pointing error (rotational) for the \( m \)th microphone. We incorporate above system errors into the farfield array model in Eq. (1), we redefine the sound pressure representation.
\[ p(x_m, \omega) = s(\omega) A_m e^{j \omega x_m} + n(x_m, \omega), \quad m = 1, 2, \ldots, M. \]  \hspace{1cm} (8)

As shown in FIGURE 1, the Monte-Carlo simulation is utilized in formulating the uniform and normal distributions to represent the combinations of array characteristics. The uniform and normal pdfs are

\[ f(x) = \begin{cases} \frac{1}{2\sqrt{3} \sigma}, & -\sigma \sqrt{3} \leq x - \mu \leq \sigma \sqrt{3} \\ 0, & x - \mu < -\sigma \sqrt{3}, \quad x - \mu > \sigma \sqrt{3} \end{cases} \]  \hspace{1cm} (9)

and

\[ f(x) = \frac{1}{\sqrt{2\pi \sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \]  \hspace{1cm} (10)

where \( \mu \) and \( \sigma \) denote the mean and standard deviation of the distribution. The notations of array characteristics can be expressed as \( A_m(\omega, \theta) \sim u(\mu, \sigma^2) \) for uniform distribution and \( A_m(\omega, \theta) \sim N(\mu, \sigma^2) \) for normal distribution. Statistics including the mean, maximum, minimum and the ML of performance measures (DI and CI) are compared with random sampling. In particular, the maximum and minimum performance measures of normal distribution are defined as \(-3\sigma\) and \(3\sigma\).

III. OPTIMAL SYNTHESIS OF BEAMFORMERS

A. Least-squares approach

The optimal beamformer design is achieved by maximizing the cost function of the array which is equivalent to the following constrained optimization problem:

\[
\begin{align*}
\text{min} & \quad w^H B w \\
\text{st.} & \quad w^H a(\theta_0) = 1
\end{align*}
\]  \hspace{1cm} (11)

for the DI cost function. In the equations, \( a(\theta_0) \) is the steering vector at the desired look direction \( \theta_0 \). For ill-conditioned \( B \), we may regularize the weight vector by diagonal loading. In practice, however, directivity is often not the only design factor that needs to be considered. Another key factor WNG concerning the implementability of an array must be taken into account in the optimization formulation.

B. Convex optimization approach

This section presents a data-independent beamformer design approach using CVX with \( \ell_1, \ell_2, \text{ and } \ell_\infty \) norms. CVX provides a versatile means to incorporate an objective function and constraints amenable to the optimization of array weights. At each frequency, we wish to determine the weight vector \( w \), as described by the following optimization problem:

\[
\begin{align*}
\text{min} & \quad \|w^H A\|_2 \\
\text{st.} & \quad w^H a(\theta_0) = 1, \quad \left( 1/\sqrt{M} \right) \leq \|w\|_p \leq \delta.
\end{align*}
\]  \hspace{1cm} (12)

\( M \) is the number of microphones. The aim of this optimization is to minimize the side-lobe level and constrain the weight two-norm. In the equation, \( \delta \) is a frequency-dependent threshold imposed on the \( \ell_2 \)-norm of the array weight vector. \( A = [a(\theta_1) \ldots a(\theta_r)] \) is a matrix with its columns comprised of manifold vectors in the rejection region. The rejection angles of CI are defined in the range \( \left( \theta_1, \theta_r \right) = (40^\circ, 180^\circ) \) with \( 10^\circ \) steps. \( \|\cdot\|_p \) denotes the vector \( p \)-norm \( (p = 1, 2, \infty) \). Two synthesis criteria of optimal beamformer including max mean DI for LS and max min CI for CVX, are utilized to investigate the performance of desired frequency range. The max mean DI is maximizing the mean deviation of DI, while the max min CI is maximizing the worst case of CI.
IV. SIMULATIONS

Simulations were undertaken to validate performance measures (DI and CI) of beamformers designed using the aforementioned LS and CVX techniques. The Matlab® solver named “cvx” developed by Grant and Boyd was employed in the work. In this simulation, we wish to design a 3-element farfield endfire linear array with inter-element spacing $d = 1$ cm for the nominal look angle $\theta = 0^\circ$ measured from the array axis. The design frequency is 1 kHz. The Mean and standard deviation ($\mu$ and $\sigma$) of the uniform pdf for system errors are defined as: gain ($\mu = 1$ and $\sigma = 0.3/\sqrt{3}$), phase ($\mu = 0$ and $\sigma = 3/\sqrt{3}$), position ($\mu = 0$ and $\sigma = d/2\sqrt{3}$) and rotational pointing ($\mu = 0$ and $\sigma = 5/\sqrt{3}$). On the other hand, the statistical parameters of the normal pdf for system errors are defined as: gain ($\mu = 1$ and $\sigma = 0.1$), phase ($\mu = 0$ and $\sigma = 1$), position ($\mu = 0$ and $\sigma = d/6$) and rotational pointing ($\mu = 0$ and $\sigma = 2$). The number of random samples is 1000. In the following, the performance of the optimal algorithms above is compared in terms of DI, CI and directivity pattern when various system errors are present. In the simulation, we will investigate gain, phase, position and pointing deviations. All system errors have combined to investigate the result for LS and CVX methods with thresholds $\epsilon$ and $\delta$, as shown in FIGURE 2. The constant $\varepsilon$ varies from zero to infinity, which corresponds to the unregularized SDA beamformer and the DAS beamformer, respectively. Due to the limitation of space, only two synthesis criteria including max mean DI for LS and max min CI for CVX are used in the simulation, respectively. In the FIGURE 2, the performance of normal distribution is better than the uniform distribution with optimal thresholds ($\epsilon$ and $\delta$). In the CVX design, there were no significant differences with pointing error between the statistics of mean and ML. FIGURE 3 show the optimal thresholds for the LS and CVX approaches at the $f=1000$ Hz. The $\ell_1$-norm design gives the lowest side lobes at the expense of the largest main lobe. The $\ell_\infty$-norm design gives the narrowest main lobe, while numerous side lobes are clearly visible. The main lobe and side lobe behavior of the $\ell_2$-norm design appears to be compromises between the $\ell_1$ and $\ell_\infty$-norm designs. Clearly visible is the different trend in DI for LS and CI for CVX designs, which entails optimal tradeoffs among the design factors. The optimal threshold of CVX design has attained higher performance than the LS design. Results have shown that four statistics have provided the optimal parameter in the LS and CVX designs which depend on the different synthesis criteria.

V. CONCLUDING REMARKS

In this paper, we presented LS and CVX design procedures for enhancing the robustness of optimal beamformers against unknown system errors. By taking into account the microphone characteristics and pointing deviation, four statistics including the mean, maximum, minimum and the ML of performance measures using MCS can be efficiently obtained the performance curves with random sampling. According to synthesis criteria of max mean DI and max min CI, it was found in this investigation that the choice of the parameters $\epsilon$ and $\delta$ are crucial to achieving the optimal compromise between the performance and robustness in the case of LS and CVX approaches. CVX provides a flexible and efficient tool in resolving the trade-off between directivity and robustness. It was observed in the simulation results of CVX approach that robust performance of the arrays against sensor mismatch and pointing error is only guaranteed when $\delta$ is sufficiently small. Inadequate choice of design parameters $\epsilon$ and $\delta$ can significantly compromise the array performance.

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REFERENCES

![Flowchart](image_url)

**FIGURE 1.** Flowchart of MC simulation of performance measures.
FIGURE 2. Compared with Mean, maximum, minimum and the ML of performance measures (DI and CI) by LS and CVX approaches. (a) Uniform distribution of system errors, (b) normal distribution of system errors.
FIGURE 3.  Spatial directivity pattern of uniform distribution of system errors at the $f=1000$ Hz.