A hybrid modeling approach for vibroacoustic systems with attached sound packages

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Modeling complex vibroacoustic systems including poroelastic materials using Finite Element (FE) based methods can be computationally expensive. Several attempts have been made to alleviate this drawback, such as high order hierarchical basis and substructuring approaches. Still, these methods remain computationally expensive or limited to simple configurations. On the other hand, analytical approaches, such as the Transfer Matrix Method (TMM), are often used thanks to the lower computational burden. However, since the geometrical flexibility of the FE method is always needed in the low/mid-frequency range, attempts have been made to couple a FE model of the elastic and acoustic domains with a TM model of the sound package. Although these hybrid approaches seem promising, the open literature is not comprehensive. The aim of this work is to present a hybrid FE-TMM approach based on a Green's function formulation. The idea is to account for the sound package by approximating the self and mutual effects on the FE nodes at the interface using fundamental solutions (i.e., Green's functions) obtained by the TMM. A benchmark representative of typical applications is used to illustrate the capabilities of the presented methodology in terms of efficiency and accuracy in comparison to other classical methods.

Published by the Acoustical Society of America through the American Institute of Physics
INTRODUCTION

Nowadays, poroelastic materials are widely used as passive treatments in aeronautic and automotive industries to obtain a reduction of the interior noise level. As a consequence, efficient and accurate tools able to predict the frequency response of systems involving poroelastic materials are needed for design purposes. This fact has led to an increasing research effort in the last two decades, mainly focused, on the one hand, on sophisticated Finite Element (FE) methodologies and, on the other hand, on fast analytical approaches. However, modeling poroelastic materials in a FE-based environment leads to severe computational issues. To alleviate this drawback, several attempts have been made. Firstly, the convergence of the FE formulation has been improved using hierarchical elements (Rigobert et al., 2003; Hörlin et al., 2001), but computational issues still remain. Then, substructuring approaches based on condensation procedures (Hamdi et al., 2000) and on modal analysis (Dazel et al., 2003, 2010) have been investigated. Recently, enrichment methods (Chazot et al., 2011) and the Wave Based Method (WBM) (Deckers et al., 2012) have been applied to poroelastic materials. Still, these methods remain computationally expensive or limited to simple configurations.

On the other side, an alternative to these sophisticated numerical approaches is the Transfer Matrix Method (TMM) (Allard and Atalla, 2009), which accounts for wave propagation in infinitely extended and flat multilayered systems. However, since a detailed description of the master systems (e.g. main structure) is always necessary for low/mid-frequency analysis, few attempts have been made to couple a FE model of the elastic and acoustic domains and a TM model of the sound package in a hybrid FE-TM model. In the hybrid methodology presented by Tournour et al. (2007), the sound package is taken into account in a locally reacting sense by the frequency dependent coefficients of the transfer matrix of the sound package itself. Approaches based on a Green’s Function (GF) formulation have been developed by Shorter and Mueller (2008) and Courtois and Bertolini (2010). Here, the basic idea is to replace the discrete GF at the interface of the actual finite size sound package with the one calculated employing the TMM of the same infinitely extended sound package. It can be pointed out that, potentially, these hybrid methodologies may be very efficient from the computational point of view compared with the previously mentioned approaches. Indeed, the sound package is entirely characterized by analytical kernels which can be evaluated quickly at each frequency employing the classical TMM. Moreover, not less important, a 3D model of the sound package is no longer needed, saving time and resources during the preprocessing phase. On the other hand, these advantages are achieved by introducing several assumptions, whose impact on the accuracy of the final hybrid model still needs to be fully assessed. The sound package is, in fact, considered flat and of infinite extent, neglecting lateral mounting conditions (i.e. sound package lateral boundary conditions) and, if present, curvature.

In this paper, a GF approach to account for sound packages is presented and its accuracy is assessed by a comparison with the FEM. A typical application is then considered to evaluate the accuracy of the assembled hybrid model. A comparison between the proposed approach, the hybrid methodology presented in (Tournour et al., 2007) and the full FE solution is systematically considered.

REVIEW OF THE METHOD

The methodology is based on a standard FE approximation of the master subsystems (i.e. elastic structures and/or acoustic cavities), which leads to the following linear system:

\[ \textbf{D} \textbf{u} = \textbf{F} + \textbf{T} \]  

where \( \textbf{D} \) is the FE matrix (e.g. dynamic stiffness of an elastic structure), \( \textbf{u} \) are the generic nodal
unknowns (e.g. structural displacements), $\mathbf{F}$ is the external load vector and $\mathbf{T}$ is the vector of the reaction forces due to the presence of the sound package. Eq. 1 can be projected onto the modal subspace to improve the efficiency of the model. The methodology proposed here uses the TMM to derive a relation between the reaction forces vector $\mathbf{T}$ and the unknowns vector $\mathbf{u}$, such that

$$\mathbf{T} = \mathbf{D}_{sp} \mathbf{u}$$

(2)

Depending on the physical dimensions of its coefficients, matrix $\mathbf{D}_{sp}$ plays the role of added dynamic stiffness, admittance and coupling to the system in eq. 1.

**Green’s Function Approach**

![Diagram](image)

**Figure 1:** 2-D and 3-D view of the unwrapped infinitely extended sound package and its interface with the FE domain.

Let $S$ be the treated area, i.e. the interface between the FE domain and the sound package. Without loss of generality, let the FE domain be an elastic solid, which is in contact with the A-side of the sound package along $S$ (see figure 1). The normal vector $\mathbf{n}$ of the solid is pointing towards the sound package. The normal stress and normal displacement at the A-side (resp. B-side) of the sound package are denoted by $\sigma_A$ and $u_A$ (resp. $\sigma_B$ and $u_B$). The work done at the interface by the normal stress $\sigma_A$ for the associated normal virtual displacement $\delta u_A$ can be written as

$$\int_S \delta u_A^* (\mathbf{x}) \sigma_A (\mathbf{x}) \, dS = \int_S \delta u_A^* (\mathbf{x}) \int_{S'} D (\mathbf{x} - \mathbf{x'}) u_A (\mathbf{x'}) \, dS' \, dS$$

(3)

where the GF notation has been introduced to write the stress $\sigma_A$ over the area $S$. The space variable $\mathbf{x}$ indicates the spatial position over the flat (or unwrapped) surface $S$. The dependency on the circular frequency $\omega$ has been omitted to simplify the notation. The superscript * indicates the conjugate of a complex scalar or the hermitian transpose of a complex vector. The analytical kernel $D (\mathbf{x} - \mathbf{x'})$ is the continuous GF of the sound package in the spatial domain, function of the distance between source (i.e. displacement at $\mathbf{x'}$) and output (i.e. normal stress at $\mathbf{x}$). Assuming the surface $S$ embedded into an infinite rigid baffle, the RHS of eq. 3 can be expressed in the wavenumber domain, yielding

$$\int_S \delta u_A^* (\mathbf{x}) \sigma_A (\mathbf{x}) \, dS = \frac{1}{(2\pi)^2} \int_R \delta \hat{u}_{\lambda}^* (\mathbf{k}) \hat{D} (\mathbf{k}) \hat{u}_{\lambda} (\mathbf{k}) \, dR$$

(4)

which is a restatement of the Parseval’s theorem for the equality of the work done in the physical and transformed domains. The notation used in eq. 4 is such that the generic complex function $\hat{f}(\mathbf{k})$ is the Fourier Transform (FT) of $f(\mathbf{x})$, while $R$ indicates the wavenumber space ($k_x, k_y$). Since eq. 4 is expressed in the wavenumber domain, the TMM can be employed to evaluate $\hat{D} (\mathbf{k})$, namely the normal stress $\delta \hat{u}_{\lambda} (\mathbf{k})$ due to a wave-like displacement such that $\hat{u}_{\lambda} (\mathbf{k}) = 1$. Therefore, the function $\hat{D} (\mathbf{k})$ is the dynamic stiffness seen from the A-side of the sound package.
The next step consists in the evaluation of the integral in the transformed space. For this purpose, the displacement $u_A(x)$ must be approximated over the surface $S$ by a superposition of simple functions. Moreover, these shape functions should be chosen such that their FT leads to a simple evaluation of the integral. Radially symmetric basis functions answer to such need. However, different choices can be found in the literature; Hassan (2007) and, later, Shorter and Mueller (2008) proposed simple piston functions, while Langley (2007) considered the use of Jinc functions. In the present work, the authors propose a different choice driven by the sake of comparison with the classical FEM. Namely, linear radially symmetric functions have been considered to approximate the behavior of the FE mesh at the interface with the sound package, i.e.

$$u(x) = \sum_i U_{A_i} \psi(x - x_i) = \sum_i U_{A_i} \psi_i(r)$$

(5)

$$\psi(r) = \begin{cases} 
1 - \frac{r}{R} & \text{if } r \leq R \\
0 & \text{if } r > R 
\end{cases}$$

(6)

where $r$ is the distance between $x_i$ and $x$. Using eq. 5, eq. 4 can be solved by collocation, yielding

$$T_{A_i} = \frac{1}{2\pi} \int_0^\infty \hat{D}(k) |\hat{\psi}(k)|^2 J_0(k r_{ij}) k \, dk \, U_{A_j} = D_{ij}^p U_{A_j}$$

(7)

where $k = |k|$ is the modulus of the wavenumber, $\hat{\psi}(k)$ is the FT of the reference shape function $\psi(r)$, $J_0$ is the zero order Bessel’s function and $r_{ij}$ is the distance between the $i^{th}$ and $j^{th}$ node. Furthermore, an isotropic sound package has been implicitly assumed, i.e. $\hat{D}(k)$ does not depend on the heading angle. Eq. 7 relates the normal force $T_A$ acting on the $i^{th}$ node with the normal displacement $U_A$ of the $j^{th}$ node (see figure 1). Therefore, $D_{ij}^p$ is the discrete GF which accounts for the effects of the sound package seen from the A-side. Finally, if the collocation points coincide with the FE nodes of the master subsystem, then, thanks to the continuity of normal forces and displacements, eq. 7 can be rewritten in matrix form as eq. 2.

To assess the validity of the proposed GF based dynamic stiffness (eq. 7), a comparison with the exact FE computation is presented. The studied configuration involves a three-layer sound package in contact with a solid over the A-side and with a pressure release condition applied on the B-side (i.e. $\sigma_B = 0$). The multilayer treatment consists, from the solid side, in a small air gap, a melamine foam and a mass layer. The latter was modeled as a solid layer with negligible Young's modulus (see table 1). The FE model of the sound package is sealed and can slide along its edges. Figure 2 shows the dynamic stiffness of the sound package for four different distances $r_{ij}$ between two internal points $x_i$ and $x_j$. In the selected frequency range, two peaks are visible: the first one, around 300 Hz, is the spring-mass resonance of the system, while the second peak, around 800-900 Hz, is related to the shear waves propagating in the poroelastic domain. Both these peaks are captured by the dynamic stiffness computed using the TM kernel $\hat{D}(k)$, although the second peak is clearly more pronounced in the FE solution. This effect is imputable to the lateral boundary conditions, which are neglected in the TMM. Overall, it can be pointed out that, for the considered system, assuming a sound package of infinite extent seems to provide a good approximation even at low frequency.

**Equivalent Locally Reacting Kernel**

The approach proposed by Tournour et al. (2007) can be obtained by simply choosing a constant value for the analytical kernel $\hat{D}(k)$ in eq. 4
The last integral in eq. 8 can be approximated by FE and assembled in the final system. This simplification allows for a very efficient implementation (i.e. the FE matrix itself is sparse and frequency independent), but the choice of $D_0$ is crucial. Typical choices consist in sampling the kernel $\hat{D}(k)$ at $k = 0$ or at the wavenumber of the external acoustic excitation (e.g. $\omega/c_0 \sin(\theta)$ for a plane wave with incident angle $\theta$ propagating in an acoustic medium with speed of sound $c_0$). Although this approach is widely used for fast computations, the example of the next section shows that this simplification can lead to a crude approximation.
RESULTS

In this section the case of a plate with an attached sound package is considered as a representative application (see figure 3). The steel plate (density $= 8000 \text{ kg/m}^3$, Young's modulus $= 200 \text{ GPa}$ and Poisson's ratio $= 0.33$) is 2 mm thick and is clamped along its four edges. No damping is provided by the plate itself. A sound package is attached onto the plate, such that the treated area $S$ coincides with the plate surface. Two boundary conditions are considered at the B-side of the acoustic treatment, namely radiation in a semi-infinite air medium and pressure release. The frequency range of interest is 10-700 Hz. The FE mesh consists in $30 \times 60$ quadrilateral elements over the $(x, y)$ plane. Concerning the FE model of the sound package, the number of brick elements used in the thickness direction is reported in table 1 together with the physical properties of each layer. A linear formulation of the elements was considered. Poroelastic elements according to (Atalla et al., 1998) were used. Three different methodologies are compared: the direct FE solution, i.e. the reference solution, the proposed hybrid GF approach and the classical hybrid methodology (i.e. locally reacting) as reviewed above.

The first analysis refers to a simple sound package consisting, from the plate side, in a thin air gap and a melamine foam (see table 1 and sound package 1 in figure 3). The B-side of the treatment is radiating into a semi-infinite air medium. The plate is excited by an oblique incidence plane wave ($45^\circ$, $45^\circ$). The value of the locally reacting kernel $D_0$ was chosen using the wavenumber of the acoustic excitation, as previously explained. Figure 4(a) shows the comparison for the quadratic velocity $<u^2>$ of the plate. The GF approach gives a perfect match with the reference FE solution. On the other side, the simplification introduced by the classical hybrid approach produces overdamping at the plate resonances. This result should not surprise the reader; indeed, taking into account for the radiation condition in a locally reacting sense typically leads to an overestimation of the damping. This phenomenon has been previously shown by Atalla et al. (2006) for a full FE model of a piece of foam. However, if the radiation condition is replaced by a pressure release condition, a very good correlation is observed for both the hybrid solutions (see figure 4(b)).

Next, we consider a three-layer treatment consisting, from the plate side, in a thin air gap, a melamine foam and an impervious mass layer (see table 1 and sound package 2 in figure 3). At the rear of the mass layer a pressure release condition is applied. The same set up was considered in the theoretical section to assess the accuracy of the dynamic stiffness evaluated by means of the TM kernel $\hat{D}(k)$. The plate is excited by a point force acting along the $z$ axis. The coordinates of the excitation point in the $(x, y)$ plane are $(0.053, 1.643)$ m. In this context, the value of the locally reacting kernel $D_0$ was chosen as $\hat{D}(k = 0)$. Figure 5(a) shows the quadratic

![Figure 3: Geometry of the treated plate system.](image-url)
velocity $<v^2>$ of the plate obtained by the hybrid models compared with the reference FE solution. Although the dynamic behavior of the sound package has been complicated by adding the heavy mass, the GF approach still captures perfectly the qualitative and quantitative behavior of the system. On the other side, the locally reacting methodology fails over almost the entire frequency range, as the effect of the mass layer on the dynamics of the plate is only partially captured. Indeed, the damping controlled region around 300Hz due to the mass-spring-mass resonance is still visible, but the dissipation is clearly overestimated. Far from this frequency range, the effect is, instead, opposite (i.e. underestimation of the damping).

Finally, it can be shown that the power radiated by the B-side of the sound package writes

$$\Pi_{\text{rad}} = \frac{\omega^2}{2(2\pi)^2} \int R \hat{u}_B^*(k) \text{Re}\{\hat{Z}_{\infty}(k)\} |\hat{D}_\phi(k)|^2 \hat{u}_A(k) dR = \frac{\omega^2}{2(2\pi)^2} U_A^* \text{Re}\{D_{ij}^{\text{rad}}\} U_{ki}$$  \hspace{1cm} (9)

where $\hat{Z}_{\infty}(k)$ is the plane wave radiation impedance and $\hat{D}_\phi(k)$ is the coupling kernel, i.e. the displacement $\hat{u}_B(k)$ produced by an imposed wave-like displacement such that $\hat{u}_A(k) = 1$. The matrix $D_{ij}^{\text{rad}}$ is the effective radiation stiffness seen by the plate. Once the hybrid FE-TM model has been solved and the displacements are known, eq. 9 can be applied to evaluate the radiated power.
power. Eq. 9 can also be used for a locally reacting kernel $D_{s0}$, yielding simply $\Pi_{\text{rad}} = D_{s0} \Pi_{\text{rad},s}$, where $\Pi_{\text{rad},s}$ is the power radiated by the velocity profile over the plate when the sound package has been removed. Concerning the full FE model, the radiated power has been calculated employing the Rayleigh’s integral. Figure 5(b) shows the radiated powers calculated by the considered approaches. Again, it can be pointed out that, even if the mass-spring-mass resonance is captured by both the hybrid solutions, only the GF approach gives a reliable result.

### Table 1: Materials used in the considered sound packages.

<table>
<thead>
<tr>
<th>Layer</th>
<th>Properties</th>
<th>Thickness</th>
<th>n of FEs</th>
</tr>
</thead>
<tbody>
<tr>
<td>Air gap</td>
<td>density = 1.21 kg/m³, speed</td>
<td>1 mm</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>of sound = 342.2 m/s</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Melamine</td>
<td>porosity = 0.99, resistivity</td>
<td>20 mm</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>= 10900 kg/m³ s, tortuosity</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>= 1.02, viscous length = 100µm</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>thermal length = 130µm</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mass</td>
<td>density = 1200 kg/m³</td>
<td>1 mm</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>Young’s modulus = 10 kPa</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Poisson’s ratio = 0.3</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>loss factor = 0</td>
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</tr>
</tbody>
</table>

### Concluding Remarks

In this paper, a hybrid methodology to account for the effect of sound packages into FE models has been presented. The approach is based on a GF formulation over the treated area of the FE domain. The accuracy of the TM kernels used to evaluate the discrete GF has been demonstrated for a representative problem, by comparison with exact FE calculations. The performances of the assembled hybrid model have been assessed for the case of a treated plate under mechanical and acoustic excitations. The hybrid solution is in very good agreement with the reference FE solution, even when the damping is entirely provided by the sound package. A substantial saving in computational time compared to the full FE model was achieved. In fact, the sound package is completely characterized by calculating the coefficients $D_{s0}$ at each frequency step. Using an adaptive quadrature algorithm to calculate these coefficients, the full matrix $D_{ij}$ in the physical domain requires, for the selected example, about 1 s per frequency step to be evaluated (data relative to a serial implementation on a Linux-based 3.4GHz Intel Core i7 system). This information is indicative, since the efficiency is affected by the required accuracy. On the other side, using the locally reacting kernel leads to a very efficient model, but the results are reliable only for simple sound packages.

### Acknowledgments

This work was funded by the Natural Science and Engineering Research Council of Canada, Bombardier Aerospace, Pratt & Whitney Canada and Bell Helicopter Textron.

### References


