1aSA9. Acoustic characterization of graded porous materials under the rigid frame approximation

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Graded porous materials are of growing interest because of their ability to improve the impedance matching between air and material itself. Theoretical models have been developed to predict the acoustical properties of these media. Traditionally, graded materials have been manufactured by stacking a discrete number of homogeneous porous layers with different pore microstructure. More recently, a novel foaming process for the manufacturing of porous materials with continuous pore stratification has been developed. This paper reports on the application of the numerical procedure proposed by De Ryck to invert the parameters of the pore size distribution from the impedance tube measurements for materials with continuously stratified pore microstructure. Specifically, this reconstruction procedure has been successfully applied to retrieve the flow resistivity and tortuosity profiles of graded porous materials manufactured with the method proposed by Mahasaranon et al. In this work, the porosity and standard deviation in pore size are assumed constant and measured using methods which are applied routinely for homogeneous materials characterisation. The numerical method is based on the wave splitting together with the transmission Green's functions approach, yielding an analytical expression of the objective function in the Least-square sense.

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INTRODUCTION

Graded porous materials present many advantages as designing optimized samples for sound insulation. These also are interesting for impedance matching at the boundaries as well as the potential size reduction for equivalent efficiency as homogeneous counterparts. There are models which can predict their acoustical characteristics\cite{1}. The well-known model by Johnson-Allard\cite{2} has been recently extended by De Ryck \textit{et al.} to predict the acoustical properties of these materials\cite{3}. The present paper focus on the application of the model proposed by Horoshenkov \textit{et al.}\cite{4} to materials with pore size stratification. This model is constructed from four parameters: the porosity $\Omega$, the tortuosity $q^2$, the static flow resistivity $R_b$ and the pore-size distribution $\phi$, the latter being the deviation of the homogenized pore geometry throughout the sample.

This model has been recently extended to encompass macroscopically-inhomogeneous porous media in Geslain \textit{et al.}\cite{5} using the state vector (Stroh) formalism and Peano series. The results were successfully compared with the absorption coefficient data obtained for a material sample which was manufactured using a novel foaming process developed by Mahasaranon \textit{et al.}\cite{6} and resulting in rigid frame porous materials with continuous pore stratification. Such materials are perfect candidate to validate macroscopically-inhomogeneous porous media models by means of inverse scattering. The results of this validation exercise can be compared to those obtained in Geslain \textit{et al.}\cite{5}.

The method using wave-splitting and invariant imbedding approach (WS-II)\cite{7, 8, 9, 3} has shown convergence retrieving parameters’ profiles along the sample’s thickness under the rigid frame assumption\cite{1}. Here, the same method is applied to the statistical pore-size distribution model\cite{4}, using real samples from the aforementioned new process\cite{6}. The steps are as follow:

- the reflected field from the selected porous material model is calculated using the WS-II method;
- the depth dependence of the non-acoustical parameters is found by minimizing the least-square difference between the calculated and the measured complex reflection coefficients via an optimized conjugate gradient routine\cite{10, 1};
- the material samples are tested in an impedance tube and the complex reflection coefficient is derived from the standard procedure from ISO 10534-2\cite{11}, using a couple of phase-matched (or phase calibrated);
- the sample orientation can be reversed and the measurement and inversion algorithm can be repeated for this sample orientation to improve the accuracy of the inversion algorithm.

It is found that both standard deviation in pore size and porosity are almost constant along the sample thickness. Therefore, the simultaneous reconstruction of the two most sensitive and remaining parameters, the flow resistivity $R_b(x)$ and the tortuosity $q(x)^2$, is performed. The computation time and convergence rate are discussed to assess the pros and cons of this characterization method.

MACROSCOPICALLY-INHOMOGENEOUS RIGID FRAME POROUS MATERIALS

The model for a rigid frame porous material which is used in this work is based on the Biot’s equations\cite{12}. It assumes that the saturating fluid (air) is much lighter than the material in the porous frame. Hence, attenuation, dispersion and viscothermal losses in the pore channels are all embedded in the effective density $\tilde{\rho}_e$ and the effective bulk modulus $\tilde{K}_e$\cite{2}.
An unidimensional case is considered throughout all the paper, with inhomogeneities and wave propagation occurring along the x-direction. The equations of motion are derived from the Euler equation and the mass conservation law. These are written in the frequency domain – with the Fourier transform convention \( P(x, t) = \int_{-\infty}^{\infty} p(x, \omega) e^{i\omega t} d\omega \) – as follows[3, 1]:

\[
\frac{d}{dx} \begin{pmatrix} p \\
\Omega(x)V \end{pmatrix} = \begin{pmatrix} 0 & i\omega \tilde{\rho}_e \\\n0 & 0 \end{pmatrix} \begin{pmatrix} p \\
\Omega(x)V \end{pmatrix},
\]

with \( \omega \) the angular frequency; \( \Omega \) the open porosity, which is the ratio of the fluid volume to the total sample volume; and \( \Omega(x)V \) the effective fluid particle velocity, or filtration velocity.

Effective sound speed and characteristic impedance in the equivalent fluid are defined as \( c_e(x, \omega) = \sqrt{\tilde{K}_e(x, \omega)/\tilde{\rho}_e(x, \omega)} \) and \( Z_e = \tilde{\rho}_e(x, \omega)c_e(x, \omega) \).

The effective density and bulk modulus are derived from the model by Horoshenkov and Swift[4] rather than from the Biot-Johnson-Champoux-Allard model[2]. The former assumes a near log-normal statistical pore size distribution within porous samples. This model is able to describe the acoustical behaviour of a large class of porous media as classical polyurethane foams, recycled materials consolidated from fibers and grains. Extending this model to macroscopic and continuous variations of the pore-size distribution yields[5]

\[
\begin{align*}
\tilde{\rho}_e(x, \omega) &= \frac{\rho_f q(x)^2}{\Omega(x)} \left[ 1 - \frac{\Omega(x)R(x)}{i\omega\rho_f q(x)F(x, \omega)} \right], \\
\tilde{K}_e(x, \omega) &= \frac{\Omega(x)}{\gamma P_0} \left[ \gamma - \frac{\rho_f q(x)^2(\gamma - 1)}{\Omega(x)\tilde{\rho}_f(x, \omega)P_0} \right],
\end{align*}
\]

wherein \( \gamma \) is the specific heat ratio, \( \rho_f \) the saturating fluid density, \( P_0 \) the atmospheric pressure and \( Pr \) the Prandtl number. \( F(x, \omega) \) is the viscosity correction function, accounting for losses and inertial effects at the pore walls. The pores are assumed cylindrical or conical along the inhomogeneities.

**DIRECT SCATTERING SOLUTION OF THE UNIDIMENSIONAL PROBLEM AT NORMAL INCIDENCE**

We consider a macroscopically 1D-inhomogeneous porous slab of thickness \( L \), which top surface is exposed to an incident plane wave. The sample is rigidly backed in an impedance tube as it is described in ISO 10534-2. The sample and the impedance tube setup are both depicted in fig. 1.

![Figure 1](https://example.com/figure1.png)

**FIGURE 1:** Slab of porous material of thickness \( L \). \( p^+ \) is the incident signal, \( p^- \) is the reflected signal. The sample is placed into an impedance tube with two flange-mounted microphones as in ISO 10534-2[11].

The direct scattering problem is to determine the reflected pressure field due to the presence of a macroscopically-inhomogeneous porous slab. This can be resolved by solving the system (1) for an incident acoustic excitation from a position \( x < 0 \). The solution is obtained through the Wave Splitting and Invariant Imbedding approach[3, 9].

The Wave Splitting is based on the exact decomposition of the pressure field \( p(x, \omega) \) into two components \( p^+(x, \omega) \) and \( p^-(x, \omega) \) corresponding to the forward and backward propagating waves in a homogeneous medium[3, 8]. The transformation is generalized for inhomogeneous media as
in the so-called “vacuum” wave splitting transformation,[3, 1, 9] which leads to
\[ p^\pm(x, \omega) = \frac{1}{2} \{ p(x, \omega) \pm Z_0 \Omega(x) V(x, \omega) \}, \]
where \( Z_0 = \rho_f c_f \) is the characteristic impedance of air, with \( c_f \) the sound speed in the free fluid.

In a general Wave Splitting transformation, the characteristic material impedance \( Z_x(x, \omega) \) should be used. However, the Invariant Imbedding technique allows a transformation based on the known characteristic impedance \( Z_0 \) of the saturating fluid instead[9]. In this approach, one considers that at the initial state, and for all \( x < L \), there is only one ambient fluid of characteristic impedance \( Z_0 \). The acoustical properties of an inhomogeneous porous material is constructed backward from \( x = L \) to \( x = 0 \). In the present formulation, the influence of each \( n^{th} \) layer of infinitesimal thickness \( dx \) added at \( x = L - ndx \) is taking into account as a perturbation of the impedance \( Z_0 \). The continuity conditions from layer to layer are implicitly accounted for.

A continuous approach is obtained in the following by taking the limit \( dx \to 0 \).

For all \( x \leq 0 \), the pressure waves \( p^+(x, \omega) \) and \( p^-(x, \omega) \) in the inhomogeneous material layer are the incident and reflected pressure waves, respectively. These are easily connected to the reflection coefficient \( R(\omega) \) and contain complete information about the complex structure inside the sample. The definition of the \( x \)-dependent reflection coefficient \( R(x, \omega) \) is the core of the invariant imbedding method: \( p^-(x, \omega) = R(x, \omega)p^+(x, \omega) \).

It follows that \( R(0, \omega) = R(\omega) \) is the usual reflection coefficient for the inhomogeneous porous material layer of thickness \( L \).

The imbedding equation for \( R(x, \omega) \) finally reads
\[
\frac{d}{dx} R = 2 A^+ R + A^- (1 + R^2).
\] (3)

This is a nonlinear Riccati differential equation, which can be solved with a classical fourth order Runge Kutta algorithm. According to the definition of the imbedding geometry, there is no porous subslab between \( x \) and \( L \) when \( x = L \). This means that there is just a rigid wall within the impedance tube so that the required boundary condition is found as \( R(L, \omega) = 1 \).

**OPTIMIZATION APPROACH OF THE INVERSE SCATTERING PROBLEM**

The inverse scattering problem consists in the retrieval of the constitutive materials parameters in (2) from the measurement of the pressure waves reflected by an inhomogeneous porous sample rigidly backed. The solution of the direct scattering problem is used to improve each profile at each step of an iterative optimization approach.

First, the four parameters to reconstruct are stacked in the vector \( \mathbf{z} = (\Omega(x), q(x)^2, R_b(x), \phi(x))^T \), where the superscript \( T \) denotes the transpose operation. An objective function \( J(\mathbf{z}) \) is defined such that[9, 1]
\[
J(\mathbf{z}) = \sum_{\omega_{\text{min}}}^{\omega_{\text{max}}} W_R(\omega)[R(0, \omega; \mathbf{z}) - R_m(\omega)]^2,
\] (4)
where \( R_m \) is the measured reflection coefficient; \( R(0, \omega; \mathbf{z}) \) is the reflection coefficient calculated at \( x = 0 \) with the help of our direct solver; and \( W_R \) is a nonnegative frequency-dependent weighting function. The objective function is defined such that a summation is performed over a frequency band of \([\omega_{\text{min}}, \omega_{\text{max}}]\), that is restricted here by the impedance tube diameter (higher limit) and signal-to-noise ratio induced by the measurement devices (lower limit).

The optimal values of the elements of \( \mathbf{z} \) are obtained when the objective function \( J(\mathbf{z}) \) is minimal. If the gradient of the objective function is known for all \( x \), then the convergence to the global minimum is good and the profiles for each parameter, along the sample thickness, are accurately reconstructed. Due to the Wave Splitting and Invariant Imbedding approach, an
analytical gradient of the cost function is deriveable, which is the most suitable to run the Conjugate Gradient method (CG) to minimize the cost function $J$ in (4) [10, 1].

**Profile reconstruction of unidimensional material parameters from impedance tube measurements**

The geometry of the impedance tube as in figure 1 is known to have two frequency limits: i) a lower limit, which is only related to the signal-to-noise ratio, to the linearity of the emitter (acoustic driver) and of the receivers (microphones), and to the isolation of the tube to the external environment and vibrations; and ii) a higher limit implied by the tube diameter as well as the microphone spacing.

For the simultaneous reconstruction of two profiles, the use of the real and imaginary parts of the reflection coefficient may be not enough. An improvement in the parameter profile reconstruction can be achieved by reversing the sample within the tube[5]. This improves convergence, when the material property gradient is relatively large.

The samples are built to be cast into an impedance tube of 29 mm diameter with a microphone spacing yielding a frequency range of 100-6400 Hz. The sample thickness is 81 mm. For the calculation via Wave Splitting and Invariant Imbedding, 200 frequency points with a logarithmic spacing between 160 and 6400 Hz are used, and the samples is discretized into 200 points, according to its physical thicknesses. These limits were chosen by looking at the coherence $\gamma = |G_{12}|^2/G_{11}G_{22}$ versus frequency, with $G_{xy} = p(\text{mic } x)^* \times p(\text{mic } y)$, $((x, y) \in [1;2]^2$ the cross-power spectrum between mic $x$ and mic $y$ (reference), where $p(\text{mic } x)^*$ is the complex conjugate of $p(\text{mic } x)$.

The weighting function in (4) is selected to reduce the influence of the higher frequency data which quality depends largely on the low frequency behaviour of the reflection coefficient. A suitable weighting function is $W_R(\omega) = 10^{-6[(\omega-\omega_{\min})/\omega_{\max}]^2}$. The solver is used with broad constraints, limited to non-negative parameters. A maximum number of iterations is needed to avoid a too high calculation time is 50. Other stopping criteria are the threshold below which the fit between the reflection coefficients is acceptable, $\epsilon = 10^{-6}$, and when the standard deviation from 5 consecutive iterations reaches a value less than $10^{-12}$.

After measurements in both direct and reverse orientations, the sample is sliced into two pieces of 40.5 mm thick. These two samples are then characterized both in direct and reversed orientations through classical methods used for homogeneous material characterisation. The values of these parameters obtained for the two samples in the in direct and reverse orientation are close with less than 10%. Table1 present the mean values of the recovered parameters obtained from these characterizations in direct and reverse orientations.

<table>
<thead>
<tr>
<th>Sample</th>
<th>$\phi$</th>
<th>$\Omega$</th>
<th>$q^2$</th>
<th>$R_b$ (N.s.m$^{-4}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 (0 - 40.5 mm)</td>
<td>0.72</td>
<td>0.94</td>
<td>1.6</td>
<td>6300</td>
</tr>
<tr>
<td>2 (40.5 - 81 mm)</td>
<td>0.75</td>
<td>0.96</td>
<td>1.2</td>
<td>3750</td>
</tr>
</tbody>
</table>

The recovery of the tortuosity $q^2$ and flow resistivity $R_b$ profiles are also performed with constant value of standard deviation in pore size and porosity $\phi = 0.74$ and $\Omega = 0.94$. The initial guesses are $q^2 = 1.5$ and $R_b = 1000$ N.s.m$^{-4}$. The reconstructed profiles and the fit between the measured reflection coefficients and the calculated ones are shown in figure 2.

In the present case, the minimizer stopped after 10 iterations (maximum iteration was fixed at 50) and the final value of the cost function, obtained after a computation time of 111 s on a regular computer. A good fit in the reflection coefficients can be observed in Figure 2(b).
CONCLUSION

The reconstruction method as detailed in De Ryck et al. [1] is successfully applied to retrieve two key material parameters from the model of Horoshenkov and Swift [4]. A new challenge here was to apply this method successfully to reconstruct the flow resistivity and tortuosity profiles in a real porous sample for which the reflection coefficient data are available over a limited frequency range. A real sample with a clearly defined pore size stratification was manufactured for this purpose and its acoustical and non-acoustical properties were accurately measured. It is possible to achieve an improvement in the accuracy of this reconstruction with a limited number of iterations by making use of the reflection coefficient data for the reverse sample orientation.

REFERENCES


