1pSA10. Reflexion of flexural waves at the end of a tapered beam of quadratic profile covered with a thin viscoelastic layer

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Flexural waves propagating in a beam can be efficiently absorbed if one extremity is tapered with a power law profile and covered by a very thin viscoelastic layer (V. Krylov, JSV 274 (2004), 605-619). Such a terminaison induces an effect known as 'the acoustic black hole effect' (ABH), which is resulting from properties of propagation of flexural wave in beams having non homogeneous thicknesses: if the thickness decreases locally, flexural waves slow down and the amplitude of the displacement field increases, leading to efficient energy dissipation if an absorbing layer is placed where the thickness is minimum (V. Georgiev et al., JSV 330 (2011), 2497-2508). Absorption of the ABH terminaison is estimated thanks to the direct measurement of the reflexion coefficient, using a wave decomposition technique. Experimental modal Analysis of a ABH beam can be performed using a high resolution technique which permits to estimate the modal density. Analysis of these experimental results is performed thanks to a model based on the finite difference method. It is shown that local transverse modes are playing an important role in the absorption properties of ABH.
INTRODUCTION

Acoustic Black Hole (ABH) effect is a passive vibration reduction technique. It is based on the fact that flexural waves in beams and plates slow down if their thickness decreases. Mironov [1] establishes a power-law relationship between local thickness \( h \) and the distance from the edge \( x \)

\[
h(x) = \alpha x^m \quad (m \geq 2)
\]

that provides an infinite travel time for a wave to reach the edge of a structure whose truncated thickness tends to zero. The flexural waves then stop propagating and never reflect back. However, practical implementations of such a power-law profile always exhibit a non-zero thickness; this is why a reflected wave always occurs. Krylov et al. [2] shows that a truncated power-law profile covered by a thin layer of damping material could overcome this effect and provide an important reduction of the theoretical reflection coefficient. Further work from Gautier et al. [3] apply the effect on elliptical plates and Georgiev et al. [4] attempts to model the ABH effect on 1D-structures and evaluate it on 1D or 2D-structures. In the first section, the capability of the ABH effect to damp vibration is shown using mobility measurement. Estimation of modal overlap factor from modal parameters obtained with the Esprit Technique is provided. A direct measurement of the reflection coefficient of ABH is also presented. The second section attempts to provide a numerical model that is correlated with the experimental results. Some conclusions are drawn in the final section.

MEASUREMENT OF THE MODAL OVERLAP FACTOR OF A ABH-BEAM

Mobility of a ABH Beam

ABH effect induces a significant added damping effect on structural vibrations. In order to show this effect, the mobility of a beam with ABH treatment is compared to the one of a reference are compared. The reference beam is a prismatic beam having a rectangular cross section and made with aluminium. Its dimensions are 0.89 m x 20 mm x 1.5 mm; The ABH beam is the same except at the extremity : a parabolic profile has been manufactured on a length of 60 mm. The thickness of the extremity is approximately 30 \( \mu \)m. This extremity is covered by a very thin layer of damping material on a length of 50 mm. For the mobility measurments, the two beams are suspended vertically. The experimental setup is made of a hammer, an accelerometer and a National Instrument acquisition card, in order to acquire the acceleration and force signals. The impact takes place on the centerline of the beam, and is supposed to excite only the flexural modes. The mobility curves on Fig. 1 show that above a certain frequency, the level of vibration is considerably reduced due to the ABH effect, as previous works showed [2, 4]. ABH effect induces a clear smoothing effect of the measured mobilities. The Modal Overlap Factor (MOF) is used in classical way to quantify such smoothing effect.

![Graph showing mobility comparison](image)

**FIGURE 1.** Compared mobilities of the reference beam (dashed blue curve) and the ABH beam (solid red curve).
Modal Overlap Factor Measurement

The Modal Overlap Factor [5] is defined by

\[ \text{MOF} = \frac{\Delta f_1}{\Delta f_2} = \eta df \]  \hspace{1cm} (1)

where \( \Delta f_1 \) is -3 dB bandwidth of a resonance, \( \Delta f_2 \) is the frequency interval between two successive resonances, \( \eta \) is the structural loss factor, \( d \) is the modal density, and \( f \) is the frequency.

Estimation of the modal parameters of the structure permits the estimation of the MOF. For this purpose, the High-Resolution ESPRIT (Estimation of Signal Parameters by Rotational Invariance Technique) [6] technique is used. The identification of parameters of the resonance peaks, such as frequencies, amplitudes and dampings is possible even in the case of high modal overlap (see [7] for details). From (1), one is able to estimate the MOF from the modal parameters on a large frequency range.

Fig. 2 presents the results obtained computing the MOF of the beams. The quantities used to compute the MOF can also be studied. The reference beam has a low MOF on the considered frequency band, because of a low modal density and a low structural damping. The ABH beam presents the same MOF a low frequencies. An unusually high MOF is reached above 1500 Hz. MOF can reach values as high as 40 %. Similar observations can be made on the modal density which is higher than the modal density for the flexural vibration of a beam and on the modal damping factors that are mostly much higher than in the reference beam.

![Comparison of MOF](image)

**FIGURE 2.** Compared Modal Overlap Factor of the reference beam ( ) and the ABH beam ( ).

Measurement of the Reflection Coefficient

While it has been studied both analytically and numerically [4, 8], the reflection coefficient of the tapered beam was not directly measured. A direct measurement is one way to characterize the efficiency of the ABH effect. Using an inverse method based on transfer functions, one is able to find the coefficients of a wave expansion of the vibratory field, which permit to compute the reflection coefficient of an extremity of the beam. Assuming an harmonic regime (convention \( e^{j\omega t} \) is supposed), the transverse displacement of the beam can be written:

\[ w(x) = A e^{-j k x} + B e^{j k x} + C e^{-k x} + D e^{k x}, \]  \hspace{1cm} (2)

where \( k \) is the flexural wavenumber and \( A, B, C \) and \( D \) the amplitudes of the propagative and evanescent flexural waves. If the position \( x \) is far enough from any discontinuity, we can assume that evanescent waves are negligible. This approximation called "far field approximation" leads to the simple form of the displacement of the beam:

\[ w(x) = A e^{-j k x} + B e^{j k x}. \]  \hspace{1cm} (3)

The scalar reflection coefficient \( R \) of the flexural propagating wave is here defined as:

\[ R = B / A. \]  \hspace{1cm} (4)
Fig. 3 shows the measured reflection coefficient for a reference beam and for a tapered beam covered with a layer of damping material. Both cases have their extremity free. The reference beam has a uniform cross section. In this case, the reflection coefficient $R$ is equal to $j$ as it is known in the literature [9]. Its amplitude modulus is one and its phase is equal to $\frac{\pi}{4}/2$. It can be seen on Fig. 3 that the reflection coefficient of the tapered beam coefficient shows a severe reduction above 1500 Hz and reach values as low as 0.2. This result is completely coherent with the one obtained in Section II. Above 1500 Hz, ABH is efficient and induces both an increase of the MOF and a decrease of the reflection coefficient. Further investigations are needed to physically explain the variation of $R$.

![Graph showing measured reflection coefficient R vs frequency (Hz)](image)

**FIGURE 3.** Measured reflection coefficient $R$ for the reference beam (blue curve) and the ABH beam (red curve).

**MODELLING THE ABH STRUCTURE**

**Model**

A model is proposed to explain the experimental MOF. Since the local wavelength at very end of the ABH profile can be of the order of magnitude of the width of the beam, it suggests a 2D (i.e. plate-like) behaviour at rather low frequencies. This is the reason why a 2D-model of the structure is built, using the Kirchhoff plate theory with a variable thickness in the x-direction; this is well described in [10]. The plate is considered under free conditions on all edges and corners. The eigenvalue problem is considered and solved using a finite difference method on a stretched grid, in order to accommodate changes in the wavelength; this method is described by Bilbao [11]. This allows to obtain the eigenfrequencies and the eigenmodes of the beams used in the experiment. The eigenfrequencies permit to estimate the modal density of the structure. Modal loss factors can be found using the local equivalent loss factor of the structure [2] and solving the problem with a complex Young modulus approach.

The equation of the free transverse vibration $W(x, y)$ of the structure in case of harmonic motion is then

$$\nabla^2\left[D(x)\nabla^2W(x, y)\right]-(1-\nu)\left(\frac{\partial^2D(x)}{\partial x^2}\frac{\partial^2W(x)}{\partial y^2}\right)-\rho h(x)\omega^2W(x, y)=0$$

where $D(x)$ is the variable complex flexural rigidity, $\rho$ the mass density, $h(x)$ the thickness and $\omega$ the angular frequency. Free boundary conditions are not written here for the sake of brevity.

**Results**

The observation of the eigenmodes shows flexural modes presenting a transverse behaviour in the ABH region; this is a proof that even if the reference structure could be well represented by a 1D-model, the ABH makes it a 2D structure. Figure 4a represents experimental and numerical modal densities for the two structures. The flexural model of the reference beam presents a good correlation with the experiment, and shows that only flexural modes have been excited. The model of ABH shows a rather good correlation with the practical structures, except in the 400-600 Hz band. It shows that torsional modes, comprised in the 2D model, have indeed been excited while hitting the centerline; this behaviour is rather surprising, but may be explained by defects of the extremity of the ABH, due to the cumbersome manufacturing process. Nevertheless, considering torsional modes for both, the modal density is slightly higher for the ABH; it is a consequence of the lower phase velocity in the decreasing thickness profile.
FIGURE 4. Modal densities of experimental reference beam ( ), the simulated reference beam, beam (Δ), the experimental ABH beam (○) and the simulated ABH beam (V). (b) Modal loss factor for the simulated reference beam (Δ) and the ABH beam (V).

Results for the modal loss factors (see Fig. 4b) show that most of the ABH modes have a more or less constant loss factor (around 2.5%), yet higher than in the uniform case (0.2%). Moreover, one can distinguish modes presenting a very high loss factor, reaching values as high as 20%. These can be explained by a localized strain energy in a region that is highly damped due to the damping material. Considering the weak increase of the modal density for this practical case of ABH, the high MOF is mainly due to these high loss factors.

CONCLUSIONS

The ABH effect is indeed an efficient passive vibration reduction method. The smoothing of mobility transfer function is observed and leads to an increase of the Modal Overlap Factor. This MOF is measured for a reference beam and for an ABH beam thanks to the ESPRIT technique. The high values of the MOF for the ABH beam are due to a higher modal density and higher values for most of the modal damping factors. Only a 2D-model is able to correctly represent the practical ABH beam, which has modes presenting a locally transverse behaviour.

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REFERENCES