Structural Acoustics and Vibration
Session 2aSA: History and Application of Constrained Layer Damping

2aSA5. Nonlinear moduli estimation for rubber-like media with local inhomogeneities elastography
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Static shear deformations of a plane-parallel layer of rubber-like material created simultaneously with the uniaxial compression are considered. The layer is fixed between the rigid plates. Displacement of one plate relative to the other resulted in shear strain of the layer. This strain could reach 0.6 of the layer thickness. At such strain effects due to the cubic nonlinearity arise. It is shown that measuring the dependence of the shear stress on the shear strain along one axis at different compression along the perpendicular axis one could determine nonlinear Landau parameters. The measurements were performed in two layers of polymeric material plastisol of 7 mm thickness with a rectangular base 8.9x8.9 cm, mounted between three aluminum plates. The upper plate was loaded with masses ranging from 0 to 25 kg and was fixed in each series of the stress-strain measurements. The values of the Landau coefficient A were measured in layers with different value of linear shear modulus.

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INTRODUCTION

The growing interest in wave processes in cubically nonlinear media is caused by the opportunity to use nonlinear effects for medical diagnostics of soft tissues. Tissue elasticity in an affected region changes drastically, which provides an opportunity to reveal pathology by measuring the local velocity and attenuation of shear waves [1]. A model taking into account dissipative processes in the simplest approximation, in which mechanical stress is represented in the form of an elastic addend directly proportional to deformation and a viscous addend directly proportional to the deformation rate (the Kelvin-Voigt rheological model), is commonly used to describe the material in such case. It is evident that in a cubically nonlinear medium an elastic addend should be written in terms of cubic parabola. This approach, however, is not applicable for the values of original material nonlinear elastic constants estimation. However, there is another approach that does provide one with such constants and describes a medium in terms of its deformation energy change. Combined with uniaxial strain application this approach was used to determine nonlinear Landau modulus $A$ in a cubically nonlinear rubber-like material here in this study.

MATERIALS AND METHODS

Considering the shear strain of a cubically nonlinear medium along one coordinate one could represent the dependence of the shear stress $\sigma$ on the strain $\varepsilon$ in the form of a cubic parabola equation: $\sigma = \mu \varepsilon + \mu \beta \varepsilon^3$, where $\mu$ is the linear shear modulus, $\mu \beta$ is the nonlinear shear modulus, $\beta$ – nonlinear coefficient.

The equation of motion for the particles of the medium:

$$\rho \ddot{u}_i = \frac{\partial \sigma_{ij}}{\partial x_j},$$

where $\rho$ and $\sigma_{ij}$ stands for the medium density and the shear stress tensor component correspondingly. In the expression of shear stress tensor

$$\sigma_{ij} = \frac{\partial e}{\partial \varepsilon_{ij}},$$

$e$ stands for the density of elastic deformation energy, that could be written as an expansion in powers of the strain tensor $\varepsilon_{ij} = \frac{\partial u_i}{\partial x_j}$ using Lame coefficients ($\lambda$ and $\mu$) and nonlinear Landau parameters ($A$, $B$, $C$):

$$e = \mu \varepsilon_{ik}^2 + \frac{\lambda}{2} \varepsilon_{il}^2 + \frac{A}{3} \varepsilon_{ik} \varepsilon_{il} \varepsilon_{kl} + B \varepsilon_{ik}^2 \varepsilon_{il} + \frac{C}{3} \varepsilon_{il}^3,$$

In [1] the linear shear modulus $\mu$ and the nonlinear coefficient $\beta$ were measured in a homogeneous rubber-like polymer plastisol. The layer of 15 mm thickness was fixed without slipping between two rigid plates. Static shear strain was created by shifting one plate relative to the other. The values obtained for the static linear modulus was $\mu = 6.7 \pm 0.4$ kPa, and for nonlinear coefficient was $\beta = 0.76 \pm 0.13$.

The dynamic shear strain was created in the resonator represented by the layer of 15 mm thickness, embedded between two plates. One of the plates (lower) was set in motion. As the resonance curves were measured at different amplitudes of the acceleration of the lower plate the dependence of the first resonance frequency on the amplitude of the acceleration of the lower plate was plotted. According to calculations using the value of a static nonlinear coefficient $\beta$ the resonance frequency should increase almost twice faster than in the experiment. The dynamic nonlinear coefficient $\beta$ that provided the minimal deviation from the experimental values was two times less than the static one and was equal to 0.35. Thus, there was no quantitative agreement between the experimentally observed effects and the results of calculations performed with the parameters determined from static measurements.
Nonlinear coefficient $\beta$ is not a parameter of the material, as depends on the method of its measuring, in contrast to the nonlinear Landau parameter $A$, obtained in [2].

It should be mentioned that in [3], where the measurement of the shear modulus was performed by means of the method of torsion vibrations, the value of the linear shear modulus was $\mu = 18.6 \pm 0.6$ kPa that is approximately 4.5 times higher than the value obtained for the same material by metal sphere indentation method. The results obtained in [1,3] indicate the possibility of mutual influence of bulk and shear elastic moduli of the material.

This paper is devoted to the measurement of nonlinear parameters of rubber-like polymer material plastisol. To determine the nonlinear parameter $A$ a method based on the acoustoelastic effect was proposed [2]. The value of the nonlinear parameter was determined by measuring the velocity of shear waves depending on the applied load. In this paper we propose to measure the dependence of the shear stress of a homogeneous layer of rubber-like medium on the deformation in static regime at different loads applied to the layer and calculate the local velocities from the measured dependence.

**EXPERIMENTAL SETUP**

A scheme of the experimental setup for measuring the nonlinear parameters of rubber-like layer by means of static deformation of this layer at various vertical loads is shown in Fig. 1. Two thin layers of rubber-like polymer (1) were located between three plane-parallel rigid plates. In the experiments the polymer material plastisol was used (the manufacturer – MF Manufacturing company, USA). Its elasticity was variable during the polymerization process by adding softener or hardener to the original polymer. The plates were made of duralumin. Thin layers of wood were glued to them, to obtain good enough adhesion of plastisol. The layers of plastisol had a thickness $h = 7$ mm and the horizontal size 89x89 mm. During the sample fabrication all the three plates were bonded at a fixed distance from each other. The resulting construction was placed inside a vessel that was filled with liquid plastisol heated to polymerization temperature of 177° C. After being cooled to room temperature the structure was extracted from the vessel, bindings were removed. The upper and lower plates were firmly secured so that they could not move in a horizontal direction during the process of shear deformation of the layers. The lower plate was rigidly secured to the table. The upper plate could move in the vertical direction. The middle plate was not secured and could be moved in the horizontal and vertical directions.

**FIGURE 1.** A scheme of the experimental setup for measuring the nonlinear parameters of rubber-like layer by means of static deformation at various vertical loads on the layer. 1 – layers of plastisol of $h = 7$ mm thickness, 2 – block, 3 – metal rope, 4 – a container filled with water, 5 and 6 – micrometer indicators.

Shear strain was obtained by applying the force acting in the horizontal direction, to the middle plate. For this purpose this end a metal rope (3) was attached to the middle plate and was thrown through the block (2). A plastic vessel (4), which was filled with water during the experiment, was attached to the free end of the rope. The ratio of the force acting on the layers from the vessel to the area of contact between the layers and the middle plate is stress $\sigma$ that is the result of the sample shift. The middle plate displacement $x$ was measured with a micrometer indicator (5). To determine the relative strain $\varepsilon$ the ratio of displacement $x$ to the layer thickness $h$ was calculated. The layers were loaded so that the maximum deformation did not exceed 60%. Under such a deformation nonlinear effects could be observed [1], and the unloading curve strictly corresponds to the loading one. To create uniaxial compression [4] weights of known mass were mounted on the upper plate. Maximum load weight was 20 kg, while the stress was 25 kPa. Under the load the layers compression occurred, which was measured with a micrometer (6). Table 1 shows the
reduction in thickness of the layers depending on the load. Compression was taken into account when calculating the strain $\varepsilon$.

**TABLE 1.** The values of the layer thickness $h$, effective $\Sigma_{eff}$ and maximal $\Sigma_{max}$ stresses at different vertical load masses $m$.

<table>
<thead>
<tr>
<th>$m$, kg</th>
<th>8.9</th>
<th>15.5</th>
<th>20.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h$, mm</td>
<td>6.77</td>
<td>6.64</td>
<td>6.4</td>
</tr>
<tr>
<td>$\Sigma_{eff}$, kPa</td>
<td>8.2</td>
<td>14.2</td>
<td>18.5</td>
</tr>
<tr>
<td>$\Sigma_{max}$, kPa</td>
<td>11</td>
<td>19.2</td>
<td>24.9</td>
</tr>
</tbody>
</table>

**OBTAINED RESULTS**

Nonlinear plots of the experimental dependencies of the shear stress $\sigma$ on the shear strain $\varepsilon$ with no load and at loads of 8.9 kg, 15.5 kg and 20.1 kg are shown in Fig. 2. Corresponding bulk stresses $\Sigma$ were 0, 11, 19.2 and 24.9 kPa, respectively. The kind of dependence is typical for rubber-like materials: as the strain increase the elasticity of the sample increase. During the measurements, the load increased in steps, the time between two successive measurements was at least a minute, so one could assume that the measurements were performed in static mode.

![FIGURE 2](image-url)

**FIGURE 2.** Nonlinear plots of the experimental dependencies of the shear stress $\sigma$ on the shear strain $\varepsilon$ with no load (♦) and at loads of 8.9 kg (▲), 15.5 kg (□) and 20.1 kg (●).

For shear deformations not exceeding 0.2 of layer thickness of the stress-strain dependence is linear. The linear shear modulus $\mu_{lin}$ measured at this plot of the dependence was the same at different vertical loads and was equal to 8.1 ± 0.1 kPa. From the measured dependence of the shear stress on the shear strain the local shear moduli were defined:

$$\mu_{loc}(\varepsilon) = \frac{\Delta \sigma}{\Delta \varepsilon}, \quad (4)$$

Then according to [2] shear waves velocity $v_s$ and nonlinear parameter $A$ could be defined by formulas:

$$v_s = \sqrt{\frac{\mu_{loc}}{\rho}}, \quad (5)$$

$$A = \frac{6\mu_{lin}}{\Sigma} (\mu_{loc} - \mu_{lin} - \Sigma). \quad (6)$$
Stress distribution is assumed to be uniform. However the boundary stress equals to zero. It can be assumed that the stress distribution corresponds to a parabolic law with a maximum $\Sigma_{\text{max}}$ in the center and zero at the boundaries.

In this paper a uniform distribution was used corresponding to the effective average $\Sigma_{\text{eff}}$, obtained by replacing the parabola with a rectangle of equal area. Values of $\Sigma_{\text{max}}$ and $\Sigma_{\text{eff}}$ for each vertical load are given in Table 1.

The obtained values of nonlinear parameter $A$ are almost independent of the strain $\varepsilon$. For each vertical load its value lies mainly between -50 and -40 kPa. The average value of the nonlinear parameter $A$ for a sample with a vertical load of 8.9 kg was found to be $-45 \pm 4$ kPa, with a load of 15.5 kg $-46 \pm 3$ kPa, with a load of 20.1 kg $-47 \pm 3$ kPa. The nonlinear parameter $A$ of plastisol was assumed to be $-45 \pm 4$ kPa.

The dependence of the nonlinear parameter $A$ at the vertical load of 15.5 kg is shown by rhombs in Fig. 3. The circles show the same dependence without taking into account the effect of compression of layers under the vertical load. In this case the average value of the nonlinear parameter $A$ was found to be $-44 \pm 3$ kPa. If the distribution of stress is assumed to be uniform not with the effective value of stress $\Sigma_{\text{eff}}$, but with the maximal value $\Sigma_{\text{max}}$, nonlinear parameter $A$ is equal to $-46 \pm 2$ kPa. The dependence corresponding to this case is shown in Fig. 3 by squares.

![FIGURE 3. The dependence of the nonlinear parameter $A$ on the relative strain $\varepsilon$ at the vertical load of 15.5 kg. The rhombs show the dependence with taking into account the effect of compression of layers for uniform distribution of stress with the effective value $\Sigma_{\text{eff}}$. The squares show the dependence for uniform distribution with the value $\Sigma_{\text{max}}$. The circles show the dependence without taking into account the effect of compression of layers.](image)

On the assumption of the expression $\sigma = \mu \varepsilon + \mu \beta \varepsilon^3$ and formula (5), one could obtain that

$$\rho \nu_i^2 = \mu + 3 \mu \beta \varepsilon^2. \tag{7}$$

Thus, the relationship between nonlinear parameter $A$ nonlinear coefficient $\beta$ could be found:

$$\beta = \frac{\left(1 + \frac{A}{6 \mu}\right) \Sigma}{3 \mu \varepsilon^2}. \tag{8}$$

Nonlinear coefficient $\beta$ depends on the nonlinear parameter $A$ and the relative deformation of the material $\varepsilon$. For example, in case of a vertical load 15.5 kg and $\varepsilon = 0.3$ the parameter $\beta = 0.59 \pm 0.06$.

**DISCUSSION**

The method proposed in this paper allowed to measure the nonlinear parameters in a layer of rubber-like material statically at different vertical loads. Special bindings were made in the produced experimental setup. It became possible with their help to achieve that while the lower plate was secured and the middle plate was released, the upper plate could move vertically, being secured in the horizontal directions. When the load was applied to the
upper plate the layers underwent a uniform compression and shear strains did not arise until the load was applied to the middle plate.

The distribution of stress in the layer under the influence of vertical load in the formula (6), which is used for calculation of nonlinear parameter \( A \), is considered to be uniform. At the same time, the boundary stress is zero. Therefore, when calculating the parameter \( A \) in this paper, stress distribution was assumed uniform with an effective mean value. In the future, the uneven distribution of stresses in the layer should be taken into account.

As the result of the measurements the following values of the linear shear modulus \( \mu \) and the nonlinear parameter \( A \) of a rubber-like polymer material plastisol were obtained: \( \mu_{\text{lin}} = 8.1 \pm 0.1 \) kPa, \( A = -45 \pm 4 \) kPa. The method of a metal sphere indentation [1,3] gives the value of the linear shear modulus for the same material as used in this paper, equal to \( 8.5 \pm 0.5 \) kPa. The resulting value of the nonlinear parameter coincides with the measured value for the gelatin-agar sample with the same rigidity in [2]. The results of this paper show that the value of the shear modulus is independent of the vertical load. The quadratic dependence of the parameter \( \beta \) on the inverse shear strain, along with internal bulk strain could cause a much overvalued effective shear modulus in [3].

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**REFERENCES**