2pSA4. Reduced models for violent bubble collapse

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Among Miguel Junger’s many contributions to acoustical science and engineering were his papers and presentations on bubble acoustics. Among his many contributions for the benefit of his colleagues at Cambridge Acoustical Associates was the mentoring of this presenter during the latter’s years of graduate study at MIT. Hence, this presentation in this session. In an evaluation of five reduced models for spherically symmetric bubble collapse and rebound [J. Appl. Phys. 112, 054910 (2012)], it was found that some recent models, which incorporate acoustic effects in both the external liquid and internal gas, did not perform as well as the long-established model by Keller and Kolodner, which incorporates such effects in the liquid but not in the gas [J. Appl. Phys. 27, 1152-1161 (1956)]. Performance was assessed through comparisons against response histories produced by finite-difference solution of the Euler equations under adiabatic conditions. Further investigation revealed that neither acoustic- nor shock-wave propagation in the gas was apparent, but that a standing wave in the gas was. This has prompted an augmentation of the Keller and Kolodner model that accounts for the standing wave. The formulation and evaluation of the augmented model is the subject of this paper.

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Introduction

For several decades, engineers and scientists have productively employed single-degree-of-freedom models to understand and exploit the behavior of violently collapsing bubbles [see, e.g., Leighton (1994), Young (1989, 2005), and Brenner, Hilgenfeldt & Lohse (2002)]. Contributions have been made by researchers and practitioners in diverse areas: underwater explosions, cavitation damage, sonochemistry, and sonoluminescence.

The analysis of bubble dynamics extends back to the pioneering work of Besant (1859) and Rayleigh (1917), who considered the following intellectual problem: in an infinite domain of ideal liquid under uniform pressure, a spherical volume of liquid is suddenly annihilated and the liquid rushes radially inward to close the cavity. A more realistic problem was then considered by Lamb (1923, 1932), who filled the cavity with gas. The result was the second-order ODE

\[ a \ddot{a} + \frac{1}{2} \dot{a}^2 = \bar{\rho} \dot{P}_a^* (t), \]  

(1)

in which each overdot denotes a time derivative, \( a(t) \) is the bubble radius, \( \bar{\rho} \) is the (invariant) mass density of the liquid, \( P_a^* (t) = P_a^i (t) = P_\text{amb} \) is the gauge pressure in the gas at the bubble radius, which in this case is considered uniform throughout the bubble, and \( P_\text{amb} \) is the (static) ambient pressure at the bubble center in the absence of the bubble. The physical idealizations underlying (1) are: the liquid is inviscid and incompressible, and the gas has zero mass density.

In the years following the appearance of (1), numerous researchers expanded the model to include other factors, such as gas vapor pressure, surface tension, liquid viscosity, and external forcing pressure [see, e.g., Leighton (1994), Eq. 4.81]. However, because our focus in this paper is on acoustic phenomena, we restrict ourselves to \( P_a^* (t) \). It’s a matter of some perplexity (and injustice) that (1) is widely referred to as the Rayleigh-Plesset equation, when in fact it was first published by Lamb in 1923 and included in Section 91a of the sixth edition of his famous Hydrodynamics (1932).

A significant limitation of (1), at least for liquids of negligible viscosity such as water, is that it doesn’t account for the energy loss observed in experimental bubble-response data. A major step in addressing this deficiency was the incorporation of liquid compressibility and thus the admittance of acoustic radiation at collapse events. The first researcher to do this was Herring (1941), who was followed by several others [see, e.g., Kirkwood & Bethe (1942), Gilmore (1952), Keller & Kolodner (1956), Keller and Miksis (1980), and Prosperetti & Lezzi (1986)].
Of the various models, the one of greatest interest here is the ODE of Keller & Kolodner (1956):

\[ a\ddot{a}(1 - \dot{a}/\bar{c}_s) + \frac{\dot{a}^2}{2\bar{c}_s}(1 - \dot{a}/\bar{c}_s) = \bar{P}_e^{-1} \left[ P_a^*(t)(1 + \frac{\dot{a}}{\bar{c}_s}) + \frac{a}{\bar{c}_s} \dot{P}_a^*(t) \right], \tag{2} \]

where \( \bar{c}_s \) is the (invariant) speed of sound in the liquid for moderate adiabatic compression and expansion about the equilibrium state. The essential mathematical difference between this equation and (1) is that the velocity potential in the liquid is taken as satisfying the wave equation rather than Laplace’s equation.

It’s to be expected that, if acoustic waves are launched in the liquid at collapse events, similar waves would simultaneously be launched in the gas. However, this was apparently not considered until Moss, Levatin & Szeri (2000) formulated a second-order ODE associated with both external and internal wave propagation. A second treatment of dual wave propagation was presented by Geers and Hunter (2002), which employed external and internal doubly asymptotic approximations [Geers & Zhang (1994)]. A third treatment was formulated by Lin, Storey, and Szeri, also in 2002; they examined non-uniform fields in the gas bubble linked to bubble-surface acceleration, obtaining a third-order ODE, in contrast to the second-order predecessors.

In a recent paper, Geers, Lagumbay, and Vasilyev (2012) evaluated the reduced-order models by comparing numerical results they produced with benchmark results for a specially designed problem. The problem was one with initial conditions that launch purely incompressible flow so that all of the responses start precisely together, with \( a(0) = a \bar{a} \) and \( \dot{a}(0) = 0 \), where \( \bar{a} \) is the equilibrium radius under ambient conditions. The benchmark results were produced by finite-difference discretization of the Euler equations for both the gas and the liquid under purely adiabatic conditions. Numerical artifacts in the benchmark results were minimized by a transformation of coordinates that yielded a stationary bubble surface, thereby avoiding the numerical inaccuracies associated with a moving boundary. The comparisons showed that the results produced by the Keller and Kolodner ODE were in better agreement with the Euler-equation results than were those produced by the other ODEs.

In addition to demonstrating the superiority of (2) over the other reduced-model equations, the Euler-equation results exhibited no signs of wave propagation in the gas. Instead, they revealed a modest heaving of an otherwise spatially invariant pressure field in the bubble. This observation was supported by a perturbation analysis of the Euler equations. In this paper, that analysis is exploited to develop a modification of (2) that reflects the departure from spatial invariance.
Adiabatic Conditions

As mentioned above, we now introduce uniform adiabatic compression and expansion of the gas, \([P^*(t) + P_{amb}]a^{3\gamma} = P_{amb}a^{3\gamma}\), where \(\gamma\) is the ratio of specific heats. Hence, (1) becomes

\[
a\ddot{a} + \frac{3}{2} \dot{a}^2 + \bar{P}_r P_{amb} \left[ 1 - (a / \bar{a})^{-3\gamma} \right] = 0, \tag{3}
\]

and (2) becomes

\[
a\ddot{a}(1 - \frac{\dot{a}}{\bar{c}_r}) + \frac{3}{2} \dot{a}^2 (1 - \frac{\dot{a}}{3\bar{c}_r}) + \frac{P_{amb}}{\bar{P}_r \bar{c}_r} \dot{a} \left[ 1 + (3\gamma - 1)(a / \bar{a})^{-3\gamma} \right] + \bar{P}_r P_{amb} \left[ 1 - (a / \bar{a})^{-3\gamma} \right] = 0. \tag{4}
\]

The most important difference between (3) and (4) is the presence of the third term in the latter, which produces energy loss at each bubble collapse and rebound.

Pressure Field in the Bubble

From the Euler and local adiabatic-equation-of-state equations, the pressure-velocity equations for the gas in the bubble are [Geers, Lagumbay, and Vasilyev (2012)]

\[
\frac{\partial p^*}{\partial r} + \bar{P}_g \left( \frac{p^* + P_{amb}}{P_{amb}} \right)^{\gamma} \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} \right) = 0, \tag{5}
\]

\[
\frac{\partial p^*}{\partial t} + u \frac{\partial p^*}{\partial r} + \gamma \left( p^* + P_{amb} \right) \left( \frac{\partial u}{\partial r} + \frac{2}{r} u \right) = 0,
\]

where \(p^*(r,t) = p(r,t) - P_{amb}\), \(u(r,t)\) is the radial velocity, and \(\bar{P}_g\) is the ambient gas density.

We now introduce the non-dimensional parameters \(r = r / \bar{a}\), \(t = (\bar{c} / \bar{a})t\) (where \(\bar{c} = \sqrt{P_{amb} / \bar{P}_r}\)), \(p(r,t) = p^*(r,t) / P_{amb}\) and \(u(r,t) = u(r,t) / \bar{c}\). With these, (5) non-dimensionalize to

\[
\frac{\partial p}{\partial r} = -\epsilon_p (p + 1)^{\gamma} \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} \right), \tag{6}
\]

\[
\frac{\partial u}{\partial r} + \frac{2}{r} u = -\frac{1}{\gamma(p + 1)} \left( \frac{\partial p}{\partial t} + \bar{P}_g \frac{\partial p}{\partial r} \right),
\]

where \(\epsilon_p = \bar{P}_g / \bar{P}_r \sim 10^{-2}\).

Let us now employ the perturbation expansions

\[
p(r,t) = p_0(r,t) + \epsilon_p p_1(r,t) + \epsilon_p^2 p_2(r,t) + \ldots, \tag{7}
\]

\[
u(r,t) = u_0(r,t) + \epsilon_p u_1(r,t) + \epsilon_p^2 u_2(r,t) + \ldots,
\]

introduce them into (6), and drop all terms involving powers of \(\epsilon_p\) higher than zero to obtain
\[ \frac{\partial p_0}{\partial r} = 0, \quad \frac{\partial u_0}{\partial r} + \frac{2}{r} u_0 = -\frac{1}{\gamma (p_0 + 1)} \frac{\partial p_0}{\partial t}, \]  

the solutions to which are, for finite \( u_0(0, t) \),

\[ p_0(r, t) = P_0(t), \quad u_0(r, t) = U_0(t) r, \]  

where, with \( \dot{P}_0(t) \equiv \frac{d\dot{P}_0}{dt} \),

\[ U_0(t) = -\frac{\dot{P}_0(t)}{3\gamma [P_0(t) + 1]} . \]  

Next, we introduce (7) into (6a), make use of (9), take \( \varepsilon_\rho P_1 << P_0 + 1 \), and then drop all terms involving powers of \( \varepsilon_\rho \) higher than one to get

\[ \frac{\partial \dot{p}_1}{\partial r} = -[P_0(t) + 1]^{1/\gamma} \left[ \dot{U}_0(t) + U_0^2(t) \right] r. \]  

Hence,

\[ p_1(r, t) = P_1(t) r^2 + F_1(t), \]  

where \( F_1(t) \) is to be determined and

\[ P_1(t) = -\frac{1}{2} [P_0(t) + 1]^{1/\gamma} \left[ \dot{U}_0(t) + U_0^2(t) \right] . \]  

### Augmented Keller-Kolodner Model

To augment (2) with terms reflecting radial dependency of the pressure field in the bubble, we write it in the non-dimensional form

\[ a \ddot{a} (1 - \varepsilon_c \dot{a}) + \frac{3}{2} \dot{a}^2 (1 - \frac{1}{3} \varepsilon_c \dot{a}) = P_a(t)(1 + \varepsilon_c \dot{a}) + \varepsilon_c a \dot{P}_a(t), \]  

where \( \dot{a} = da / dt, \quad \ddot{a} = d^2a / dt^2, \quad P_a(t) = P^*/P_{am}, \quad \dot{P}_a = d\dot{P}_a / dt, \) and \( \varepsilon_c = \bar{c} / \bar{c}_f \sim 10^{-2} \). This ODE agrees, through \( O(\varepsilon_c) \), with the corresponding ODEs of Herring (1941), Kirkwood & Bethe (1942), Gilmore (1952), and Prosperetti & Lezzi (1986). From the previous section, \( P_a(t) = P_0(t) + \varepsilon_\rho [a^2 P_1(t) + F_1(t)] \), so using this in (14) and retaining terms only in \( \varepsilon^0 \) and \( \varepsilon' \), we obtain

\[ a \ddot{a} (1 - \varepsilon_c \dot{a}) + \frac{3}{2} \dot{a}^2 (1 - \frac{1}{3} \varepsilon_c \dot{a}) = P_0(t)(1 + \varepsilon_c \dot{a}) + \varepsilon_c a \dot{P}_0(t) + \varepsilon_\rho [a^2 P_1(t) + F_1(t)]. \]  

Then, introducing (10) into (13), we find

\[ P_1(t) = \frac{[P_0(t) + 1]^{1/\gamma}}{6\gamma [P_0(t) + 1]} \left[ \dot{P}_0(t) - \left( \frac{3\gamma + 1}{3\gamma} \right) \frac{\dot{P}_0^2(t)}{P_0(t) + 1} \right]. \]  

Now the pressure field in the gas is no longer assumed constant, so we employ the local adiabatic relation \( \rho_g \dot{p}_g(r, t) = \left( \frac{p_0^*}{P_{am}} \right) \dot{p}_g \rho_g(r, t) \). Hence, expanding \( p(r, t) = \rho_g \dot{p}_g(r, t) \)
as in (7), we obtain the non-dimensional relation
\[ \rho_0(r,t) + \varepsilon_\rho P_0(r,t) + O(\varepsilon_\rho^2) = \left\{ P_0(t) + \varepsilon_\rho [P_1(t) r^2 + F_1(t)] + O(\varepsilon_\rho^2) + 1 \right\}^{1/\gamma}. \] (17)

A binomial expansion of the right side of this equation yields
\[ \left\{ P_0(t) + \varepsilon_\rho [P_1(t) r^2 + F_1(t)] + O(\varepsilon_\rho^2) + 1 \right\}^{1/\gamma} = [P_0(t) + 1]^{1/\gamma} + \varepsilon_\rho \gamma^{-1} \frac{P_1(t) r^2 + F_1(t)}{[P_0(t) + 1]^{1-1/\gamma}} + O(\varepsilon_\rho^2). \] (18)

Hence, solving (17) through \( O(\varepsilon_\rho^0) \) and then again through order \( O(\varepsilon_\rho^1) \), we get
\[ \rho_0(r,t) = [P_0(t) + 1]^{1/\gamma}, \quad P_0(t) = [P_0(t) + 1]^{1-1/\gamma} \] \[ \rho_0(r,t) = \gamma^{-1} \frac{P_1(t) r^2 + F_1(t)}{[P_0(t) + 1]^{1-1/\gamma}}. \] (19)

Now conservation of mass requires, through \( O(\varepsilon_\rho) \), that
\[ 4\pi \int_0^a [\rho_0(r,t) + \varepsilon_\rho P_0(r,t)] r^2 \, dr = \frac{4}{3} \pi. \] (20)

Hence, (20) and (19) yield
\[ [P_0(t) + 1] a^{3\gamma} = \left\{ 1 - \varepsilon_\rho \gamma^{-1} \frac{3}{3} a^5 P_1(t) + a^3 F_1(t) \right\}^{\gamma}. \] (21)

A binomial expansion, through \( O(\varepsilon_\rho) \), of the right side of this equation then gives
\[ P_0(t) = a^{-3\gamma} - 1 - \varepsilon_\rho a^{-3\gamma} \frac{3}{3} a^5 P_1(t) + a^3 F_1(t) \] \[ [P_0(t) + 1]^{1-1/\gamma}. \] (22)

Solving this through \( O(\varepsilon_\rho^0) \), we get
\[ P_0(t) = a^{-3\gamma} - 1. \] (23)

Then solving it through \( O(\varepsilon_\rho^1) \), we get
\[ F_1(t) = -\frac{3}{2} a^2 P_1(t). \] (24)

To relate \( P_1(t) \) to \( a \), we use (23) in (16) to obtain the compact result
\[ P_1(t) = -\left( \frac{1}{2} a^{-5} \right) \dot{a}. \] (25)

Introducing this and (24) into (15), we get
\[ a \ddot{a} (1 - \varepsilon_c \dot{a} + \frac{1}{2} \varepsilon_\rho a^{-3}) + \frac{3}{2} \dot{a}^2 (1 - \frac{1}{2} \varepsilon_c \dot{a}) = P_0(t) (1 + \varepsilon_c \dot{a}) + \varepsilon_c a P_0(t), \] (26)

which may be compared with (14), inasmuch much as the Keller-Kolodner model makes no distinction between \( P_a(t) \) and \( P_0(t) \). With \( P_0(t) \) given by (23), (26) becomes
\[ a \ddot{a} (1 - \varepsilon_c \dot{a} + \frac{1}{2} \varepsilon_\rho a^{-3}) + \frac{3}{2} \dot{a}^2 (1 - \frac{1}{2} \varepsilon_c \dot{a}) + \varepsilon_c [1 + (3\gamma - 1) a^{-3\gamma}] + (1 - a^{-3\gamma}) = 0. \] (27)
Now from (7a), (9a), (12), (24), and (25), the non-dimensional pressure field in the gas is given, through $O(\varepsilon_p)$, by

$$p(r, t) = a^{-3\gamma} - 1 + \varepsilon_p a^{-2} \ddot{a} \left[ \frac{3}{10} - \frac{1}{2} (r / a)^2 \right].$$  \hfill (28)

From this equation and (23), we see that $p(0, t) = P_0(t) + \frac{3}{10} \varepsilon_p a^{-2} \ddot{a}$ and $p(a, t) = P_0(t) - \frac{2}{5} \varepsilon_p a^{-2} \ddot{a}$. $\ddot{a}(t)$ is typically calculated during a numerical solution, so $p(r, t)$ is readily obtained.

A less accurate alternate to (28) may be obtained by writing (27) only through $\varepsilon^0$, yielding

$$a^{-2} \ddot{a} = a^{-3}(a^{-3\gamma} - 1) - \frac{3}{2} a^{-3} \dot{a}^2.$$  \hfill (29)

Hence, (28) gives

$$p(r, t) = (a^{-3\gamma} - 1) \left[ 1 + \varepsilon_p a^{-3} \left[ \frac{3}{10} - \frac{1}{2} (r / a)^2 \right] \right] - \frac{3}{2} \varepsilon_p a^{-3} \dot{a}^2 \left[ \frac{3}{10} - \frac{1}{2} (r / a)^2 \right].$$  \hfill (30)

Now $\ddot{a}$ appears in every term of the derivative of the right side of this equation, so $\dot{p} = 0$ when $\ddot{a} = 0$. Also, $a = a_{\min}$ when $\ddot{a} = 0$. Hence, (30) produces the approximate result

$$p(r, t_p) = (a_{\min}^{-3\gamma} - 1) \left[ 1 + \varepsilon_p a_{\min}^{-3} \left[ \frac{3}{10} - \frac{1}{2} (r / a_{\min})^2 \right] \right]$$  \hfill (31)

where $t_p$ is the time of peak pressure. This result is not particularly valuable because one is still required to solve (27) to determine $a_{\min}$, which solution would produce $p(r, t_p)$ from (28) as well.

Finally, we express (27), (28), and (31) in dimensional form:

$$a \ddot{a} \left[ 1 - \frac{\ddot{a}}{\dddot{a}} \right] + \frac{P_g}{\rho_l \dddot{a}} \left( \frac{a}{\bar{a}} \right)^{-3} + \frac{3}{2} \dot{a}^2 \left( 1 - \frac{\dot{a}}{\ddot{a}} \right) + \frac{P_{amb}}{\rho_{l} \dddot{a}} \left[ 1 + (3\gamma - 1) \left( \frac{a}{\bar{a}} \right)^{-3\gamma} \right] = 0,$$

$$p^* (r, t) = P_{amb} \left[ (a / \bar{a})^{-3\gamma} - 1 \right] + \frac{P_g}{\rho_l} \left( \frac{a}{\bar{a}} \right)^{-3} a \ddot{a} \left[ \frac{3}{10} - \frac{1}{2} (r / a)^2 \right],$$  \hfill (32)

$$p^* (r, t_p) \approx P_{amb} \left[ (a_{\min} / \bar{a})^{-3\gamma} - 1 \right] \left[ 1 + \left( \frac{P_{l}}{\rho_{l}} \right) (a_{\min} / \bar{a})^{-3} \left[ \frac{3}{10} - \frac{1}{2} (r / a_{\min})^2 \right] \right].$$

The only difference between (32a) and (4) is the inclusion of a third term in the bracketed expression multiplying $a \ddot{a}$. The uniform pressure field pertinent to (4) is given by (32b) with $\rho_g = 0$.

**Numerical Comparisons**

Numerical results produced by (4) [KK equation] and (32a) [KKG equation] were compared with corresponding results produced by (5) [EE equations]. Bubble-radius, surface-velocity, and pressure histories produced by the different equations are virtually indistinguishable at macro scale. More significant differences appear in peak-pressure values occurring within a narrow time window centered at first collapse/rebound. Ratios of these values appear in Table 1, with a
circumflex denoting peak pressure. Included in the table are corresponding values for the third-order ODE of Lin, Storey, and Szeri (2002), as their equation was derived on the basis of inertial effects in the gas, which here is incorporated in the third term multiplying $a\ddot{a}$ in (32a).

### Table 1

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\hat{p}<em>{\text{const}}^{KK}/\hat{p}</em>{r=0}^{EE}$</th>
<th>$\hat{p}<em>{r=0}^{KK}/\hat{p}</em>{r=0}^{EE}$</th>
<th>$\hat{p}<em>{\text{const}}^{KK}/\hat{p}</em>{r=a}^{EE}$</th>
<th>$\hat{p}<em>{r=a}^{KK}/\hat{p}</em>{r=a}^{EE}$</th>
<th>$\hat{p}<em>{r=a}^{LSS}/\hat{p}</em>{r=a}^{EE}$</th>
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<td>0.98</td>
<td>1.05</td>
<td>1.08</td>
<td>1.04</td>
<td>1.21</td>
</tr>
<tr>
<td>4</td>
<td>0.99</td>
<td>1.20</td>
<td>1.28</td>
<td>1.16</td>
<td>2.03</td>
</tr>
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</table>

Comparing the first pair of pressure-ratio columns, we see that the KK equation slightly underestimates and the KKG equation somewhat overestimates peak pressure at the bubble center. Comparing the second pair of pressure-ratio columns, we see that the KK equation considerably overestimates and the KKG equation somewhat overestimates peak pressure at the bubble surface. Because the KK equation models the pressure in the gas as independent of the bubble radius, its accurate predictions at the bubble center are necessarily accompanied by overpredictions at the bubble surface. The last column in the table shows the LSS equation to be the least accurate of the three considered.

Volume-averaged and center-to-surface peak-pressure ratios appear in Table 2. The KK and KKG average-pressure ratios are quite close, even though the EE and KKG center-to-surface ratios show appreciable pressure variation for $\alpha = 4$. The last two columns show that center-to-surface ratios calculated from minimum-radius values exceed their counterparts calculated from surface accelerations.

### Table 2

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\hat{p}<em>{\text{avg}}^{KK}/\hat{p}</em>{\text{avg}}^{EE}$</th>
<th>$\hat{p}<em>{\text{avg}}^{KKG}/\hat{p}</em>{\text{avg}}^{EE}$</th>
<th>$\hat{p}<em>{r=0}^{EE}/\hat{p}</em>{r=a}^{EE}$</th>
<th>$\hat{p}<em>{r=0}^{KKG}/\hat{p}</em>{r=a}^{KKG}$</th>
<th>$\hat{p}<em>{r=0}^{KKG}/\hat{p}</em>{r=a}^{KKG}$ **</th>
</tr>
</thead>
<tbody>
<tr>
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<td>1.04</td>
<td>1.05</td>
<td>1.10</td>
<td>1.11</td>
<td>1.12</td>
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<tr>
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<td>1.18</td>
<td>1.29</td>
<td>1.34</td>
<td>1.39</td>
</tr>
</tbody>
</table>

* From (28) or (32b)  ** From (31) or (32c)

In conclusion, the KK model is seen to be surprisingly accurate, even for substantial pressure variation inside the bubble. The principle advantage of the KKG model is that it provides insight into that variation at minimal computational cost.
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