ICA 2013 Montreal
Montreal, Canada
2 - 7 June 2013

Structural Acoustics and Vibration
Session 3pSAa: Acoustic Metamaterials II

3pSAa6. Equations for energy characteristics of oscillatory systems with internal (hidden) degrees of freedom and application to acoustic metamaterials

Yuri Bobrovnitskii*

*Corresponding author’s address: Theoretical and applied acoustics, Mechanical Engineering Research Institute, 4, Griboedov str., Moscow, 101990, Moscow, Russia, yuri@imash.ac.ru

General equations are derived for calculating the kinetic and potential energies and other energy characteristics of linear oscillatory NDOF-systems a portion of DOFs of which are internal or inaccessible for measurement and excluded from consideration. The energy characteristics are expressed through parameters pertaining only to the input or accessible DOFs. The equations are based on the certain novel properties of the so-called Shur matrix complement. The theory is applied to calculating the energy characteristics of acoustic metamaterials for which this is still an unsolved problem, especially for those with negative effective density and stiffness. A metamaterial is thought as a medium or periodic structure in which the role of effective inertia and elastic elements is played by sufficiently complex oscillatory systems with internal DOFs. Applying the derived equations to a cell of periodicity of a metamaterial one can obtain the exact values of the needed energy characteristics. The theory is verified in computer simulation and laboratory experiment.

Published by the Acoustical Society of America through the American Institute of Physics
Most metamaterials are designed as periodic structures in which cells of periodicity are oscillatory systems having many degrees of freedom. Some of the cell DOFs are the inputs through which the cell is connected to the neighboring cells. The remaining DOFs are internal or inaccessible (hidden) and excluded from consideration. The cell is, hence, characterized by inertial and elastic parameters determined only at the inputs. These parameters are called effective parameters (other terms are dynamic or equivalent or apparent parameters). Because of the influence of the hidden DOFs, the values of the effective mass and stiffness can be positive and negative, zero and infinite. The problem is how to calculate the energy characteristics of the system using only the effective parameters.

For example, in Fig.1 2DOF-system is shown which is used in many publications as “negative effective mass”. The first mass \( m_1 \) is the single input of the system, while the tuned-mass damper \((m_2, k_2)\) represents the internal DOF.

The effective mass of the system, i.e. the ratio of the complex amplitude of the external force to the complex amplitude of the first mass acceleration, is equal to

\[
m_{\text{eff}} = \frac{f}{a} = m_1 + \frac{m_2}{1 - \frac{\omega_2^2}{\omega^2}}, \quad \omega_2^2 = \frac{k_2}{m_2}, \tag{1}
\]

It has negative values in the certain frequency band. The total energy of the system cannot obviously be calculated here as \( m_{\text{eff}} |v_1|^2/2, |v_1| \) being the velocity amplitude of \( m_1 \), since the energy must always be positive.

What is made in this presentation is a set of new rather general equations that allow one to calculate the necessary energy characteristics using only the effective parameters, i.e. parameters pertaining to the inputs. For the system in Fig.1, these equations give

\[
T = \frac{|v_1|^2}{8\omega} \frac{d}{d\omega} \left( \omega^2 m_{\text{eff}} \right),
\]

\[
U = \frac{\omega |v_1|^2}{8} \frac{dm_{\text{eff}}}{d\omega},
\]

\[
E = T + U = \frac{|v_1|^2}{4} \frac{d}{d\omega} \left( \omega m_{\text{eff}} \right). \tag{2}
\]

As is seen, the energy characteristics depend on the effective mass as well as on its derivative with respect to frequency. Equations in form (2) are valid not only for the 2DOF system in Fig.1 but for any NDOF- system with one input that plays the role of an effective mass. In this general case, the effective mass is equal to \( m_{\text{eff}} = -\text{Imag}(Z_{\text{in}}) \), where \( Z_{\text{in}} \) is the input impedance of the NDOF- system.

If such NDOF-system is used as a spring, its effective stiffness is \( k_{\text{eff}} = \omega \text{Imag}(Z_{\text{in}}) \) and the equations for energies are

\[
T = \frac{|v_1|^2}{8\omega} \frac{dk_{\text{eff}}}{d\omega}, \quad U = -\frac{\omega |v_1|^2}{8} \frac{d}{d\omega} \left( \frac{k_{\text{eff}}}{\omega^2} \right), \quad E = -\frac{|v_1|^2}{4} \frac{d}{d\omega} \left( \frac{k_{\text{eff}}}{\omega} \right). \tag{3}
\]

It is worth noting that equations of type (2), (3) allow one to obtain the energy characteristics of any linear structure using the data measured only at one point. The author with colleagues has verified it in laboratory experiments on simple structures – beams and plates in bending. The values of the kinetic and potential energies computed by equations (2) and (3) through the measured input impedance were practically indiscernible from those directly measured on the whole structures.

Consider now a 1D acoustic metamaterial or medium (metafluid) that is constructed as a periodic array of NDOF-cells. At low frequencies the array behaves like a real continuous acoustic medium. The term “acoustic”
means that there exists only one type of normal waves in the medium. For the array it means that its NDOF-cell has only two inputs. This statement follows from the known result\(^1,2\) that the number of normal wave types in a continuous medium or in a periodic structure is equal to the number of independent physical mechanisms of energy exchange between adjacent parts of the medium or adjacent cells of the periodic structure. Another known result obtained by I. Newton in 1686\(^1\) states that a 1D acoustic medium can be modeled as a simple mass-spring chain. Hence, the array of NDOF-cells can also be represented as a Newton chain; and the cell NDOF-system with two inputs can be represented as a cell containing only two elements – effective mass \(m_{\text{eff}}\) and a spring with effective stiffness \(k_{\text{eff}}\) (Fig.2). These effective parameters are derived from equality of the input responses \((v_1, v_2)\) of the two cell structures to external forces \((f_1, f_2)\):

\[
m_{\text{eff}} = -\frac{1}{\omega} \text{Imag}(Z_{11} + Z_{22} + 2Z_{12}), \quad k_{\text{eff}} = -\omega \text{Imag}(Z_{12}),
\]

where \(Z_{ij}\) are elements of the input impedance matrix of the NDOF-cell:

\[
\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.
\]

Using the Newton’s model parameters it is easy to obtain the effective parameters of the continuous medium under consideration: in its wave equation

\[
K_{\text{eff}} \frac{d^2p}{dx^2} + \rho_{\text{eff}} \omega^2 p = 0
\]

the effective bulk modulus and density should be equal to \(K_{\text{eff}} = k_{\text{eff}} l, \rho_{\text{eff}} = m_{\text{eff}} l, l\) being the length of the cell of periodicity.

The general energy equations (derived below) give in this case the following equation for the kinetic and potential energies of the NDOF-cell of the array, expressed through its effective parameters (4):

\[
T = U = \frac{E}{2} = \frac{m_{\text{eff}} |v|^2}{4} \left[ 1 + \frac{\omega}{2} \left( \frac{1}{m_{\text{eff}}} \frac{dm_{\text{eff}}}{d\omega} - \frac{1}{k_{\text{eff}}} \frac{dk_{\text{eff}}}{d\omega} \right) \right] = \frac{m_{\text{eff}} |v|^2 \omega d}{4} \left( \ln \frac{\omega_2}{\omega_0} \right), \quad \omega_2 = \frac{k_{\text{eff}}}{m_{\text{eff}}},
\]

It is assumed here that a normal wave \(v(n) = \exp(i\mu n - i\omega t)\) with a real propagation constant \(\mu\) propagates along the array. Again, the energy characteristics depend on the derivatives of the effective parameters with respect to frequency.
Several important features of metamaterials, especially of the so-called double negative metamaterials, follow from equation (5), or more exactly, from positivity of the kinetic and potential energies. Among new materials, double negative metamaterials, having negative effective density and stiffness, are the most interesting for practical application. They support waves with the phase and group velocities of opposite directions and therefore demonstrate such properties as negative refraction, inverse Doppler effect and other. From Eq.(5) follows that a double negative linear passive metamaterial with constant (independent on frequency) effective parameters cannot be constructed. This is due to the kinetic and potential energies have the sign of $m_{eff}$, where $m_{eff}$ and $k_{eff}$ are constant. One more corollary from (5) concerns the so-called complementary media. By definition, two media are complementary if their effective parameters have identical absolute values and opposite signs. The corollary is that the complementary media can exist only at a single frequency. This is because, according to equation (5), they must have identical $\omega_0$ and the derivatives $d\omega_0/d\omega$ of opposite signs. The present author tried the equation (5), as well as equations (2) and (3), on several metamaterials found in the literature and made sure that they work well.

Now some words about the general energy equations from which the particular equations (2), (3) and (5) have been obtained, and about their rigorous derivation. Consider a linear oscillatory system with N DOFs, e.g. N-mass FE-model of arbitrary discrete or continuous elastic structure. It is described by the equations

$$M\ddot{u}(t) + R\dot{u}(t) + Ku(t) = f(t),$$

where $u$ and $f$ are $N$-vectors of displacements and external forces, $MR$, and $K$ are real symmetric inertial, damping and stiffness square matrices. For harmonic motion, $f(t) = \exp(-i\omega t)$, and $u(t) = \exp(-i\omega t)$, these equations are equivalent to the matrix equation ($\nu = \dot{u}$)

$$f = Z\nu, \quad Z = R + i\left(\frac{1}{\omega}K - \omega M\right) = R + iX. \quad (6)$$

If the full inputance $N\times N$-matrix $X$ and velocity vector are known, the kinetic, potential and total energy of the system can be computed as

$$T = -\frac{1}{8}\nu^*\left(\frac{dX}{d\omega} + \frac{1}{\omega}X\right)\nu, \quad U = -\frac{1}{8}\nu^*\left(\frac{dX}{d\omega} - \frac{1}{\omega}X\right)\nu, \quad E = -\frac{1}{4}\nu^*\frac{dX}{d\omega}\nu. \quad (7)$$

These equations directly follow from (6), asterisk denotes the Hermitian conjugate. Suppose now that $N_1$ DOFs of the system are inputs and the remaining $N-N_1$ DOFs are internal. Excluding in (6) velocities of all internal DOFs, one can obtain the input impedance matrix $Z_\alpha$

$$f_\alpha = Z_\alpha \nu_\alpha, \quad Z_\alpha = Z_{\alpha\alpha} - Z_{\alpha\beta}Z^{-1}_{\beta\beta}Z_{\beta\alpha}.$$

Here $\alpha$ and $\beta$ denotes indexes $1,2,...,N_1$ and $N_1+1,...,N$. The $N_1\times N_1$-matrix of the input impedances is also called the Schur complement of the full impedance matrix $Z$ in (6). The Schur complement is a symmetric square matrix possessing many useful properties. For the purpose of this presentation, important are the following properties

$$\nu^*X\nu = \nu^*_\alpha X_{\alpha\alpha}\nu_\alpha, \quad (8)$$
$$\nu^*\frac{dX}{d\omega}\nu = \nu^*_\alpha X_{\alpha\alpha}\frac{d\nu_\alpha}{d\omega}. \quad (9)$$

Property (8) is valid for any symmetric complex impedance matrix, i.e. to any linear vibratory system. Property (9) is valid only for certain classes of matrices of which the most important is the class of symmetric pure imaginary matrices which correspond to linear lossless vibratory systems. Using these properties one can express the energy characteristics (7) through the input data.

When applied to an NDOF-system with one input, the equations give the energy characteristics through the input impedance and, after representing the system by the effective mass or stiffness, result in equations (2) and (3). When applied to an NDOF-system with two inputs which is a cell of a periodic array, one comes to equations (5). This general approach allows one analogously to obtain the necessary energy equations for more complex metamaterials, e.g. for elastic ones, that have more than one type of normal waves.

**SUMMARY**

Equations are derived for calculating the kinetic and potential energies of linear NDOF oscillatory systems using the data pertaining to a part of DOFs.
The derived equations are applied to acoustic metamaterials. Some general restrictions on metamaterials are formulated.

ACKNOWLEDGMENT

The work is supported in part by RFBR Grant No. 12-02-00222-a.

REFERENCES