4aSA1. Analysis of the acoustic scattering from a submerged bilaminar plate

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The acoustic scattering from a submerged finite bilaminar rectangular elastic plate is modeled using the exact theory of three-dimensional elasticity. The two layers of the composite bilaminar plate have the same lateral dimensions, but have different thicknesses and material properties. The plate is set in an infinite rigid baffle and is coupled to a different acoustic medium on its two surfaces. The plate is insonified by an acoustic plane wave. The farfield backscattered and forward scattered waves are computed for various layer thicknesses and elastic material properties in contact with air or water. First, the scattering from a uniform finite, baffled, steel plate elastic plate was computed. A bilaminar plate was also analyzed with one layer made of steel and the other made of a damped elastomer. The backscattered and forward scattered acoustic farfield spectra versus frequency were computed at the on-axis point receiver due to a normally incident plane wave. The directivity functions for a normally incident plane wave insonifying the steel- or elastomer- side of the plate were computed for a range of frequencies, with either water or air backing.

Published by the Acoustical Society of America through the American Institute of Physics
INTRODUCTION

The back- and forward-scattering from infinite multi-layered plates were discussed by many authors. In most of these, the elastic plates were modeled by the classical Kirchhoff or the Mindlin plate theory [1-5]. The scattering from infinite elastic plates with a line discontinuity using Mindlin plate theory were discussed by few authors [6-8]. In this paper, a rectangular bilaminar plate consists of two perfectly bonded elastic plates, each having a different thickness and material properties. The bilaminar plate’s two layers are modeled by elasticity theory. The bilaminar plate is set in an infinite rigid baffle and is in contact with a different fluid medium on each of its faces. The bilaminar plate is insonified by an acoustic plane wave applied to either of the two faces of the plate. The boundary conditions of the bilaminar plate are: stress-free as well as vanishing in-plane displacements on its periphery. This means that the entire bilaminar plate is free to move in the normal direction in response to acoustic surface pressures.

1. THEORY OF ELASTIC PLATES IN 3-D ELASTICITY

Elastic waves propagating in 3-D elastic plates satisfy the vector PDE for wave propagation in an isotropic elastic medium, i.e.

\[ \mu \nabla^2 \vec{u} + (\lambda + \mu) \nabla (\nabla \cdot \vec{u}) = \rho \frac{\partial^2 \vec{u}}{\partial t^2}, \tag{1} \]

where \( \mu \) and \( \lambda \) are the Lame’ elastic constants, and \( \rho \) is the plate material density. The vector wave solution can be obtained by using the Helmholtz decomposition in terms of three elastic wave potentials for the vector displacement field

\[ \vec{u} = \nabla \Phi + \vec{N} + \vec{M}, \]
\[ \vec{N} = \frac{1}{\beta} \nabla \times (\nabla \times \vec{e} \chi), \quad \vec{M} = \nabla \times \vec{e} \psi, \tag{2} \]

where the direction \( z \) is chosen as the coordinate normal to the plate surface. Here \( \Phi \) represents the dilatational wave potential, and \( \chi \) and \( \psi \) represent the two shear wave potentials. Assuming \( e^{i\omega t} \) time dependence throughout this paper, the three potentials satisfy their respective scalar Helmholtz wave equations

\[ \nabla^2 \Phi + \alpha^2 \Phi = 0, \quad \nabla^2 \chi + \beta^2 \chi = 0, \quad \nabla^2 \psi + \beta^2 \psi = 0, \tag{3} \]

where \( \alpha \) and \( \beta \) represent the dilatational and shear wave numbers, respectively, defined by

\[ \alpha = \frac{\omega}{c_d} \quad \beta = \frac{\omega}{c_s}, \]
\[ c_d = \left[ (\lambda + 2\mu)/\rho \right]^{\frac{1}{2}} \quad c_s = \left( \frac{\mu}{\rho} \right)^{\frac{1}{2}}, \tag{4} \]

where \( \omega \) is the circular frequency, and \( c_d \) and \( c_s \) are the dilatational and shear wave speeds, respectively.

2. PLATE GEOMETRY AND BOUNDARY CONDITIONS

Consider a rectangular plate, where the \( x \) and \( y \) coordinates lie in the plane of the plate, and the \( z \) coordinate being normal to the plate surface. A finite elastic bilaminar plate is made up of two plates whose lateral dimensions (\( a \) and \( b \)) are identical but their thicknesses (\( h_1 \) and \( h_2 \)) are different, see Fig. 1. Each plate has a different set of physical properties, i.e. densities, \( \rho_1 \) and \( \rho_2 \), Lame’ constants \( \lambda_1, \mu_1 \) and \( \lambda_2, \mu_2 \). The two plates are totally bonded at their common interface, and the bilaminar plate is pinned in the plane of the plate, so that the displacements normal to the perimeter (either \( u_x \) or \( u_y \)) vanish. The bilaminar plate is set in an infinite rigid baffle, and each plate is in contact with a different acoustic medium and is excited to vibration by an incident acoustic pressure wave as shown in Fig. 1. The two semi-infinite acoustic media #1 and #2 satisfy the scalar Helmholtz wave equation.
Furthermore, the bilaminar plate is free of shear stresses along the same perimeter. These boundary conditions can be enumerated as follows:

\[ \begin{align*}
\tau_{xy}^{1,2} \big|_{y=0,a} &= 0 \\
\tau_{xy}^{1,2} \big|_{y=0,b} &= 0 \\
\tau_{yz}^{1,2} \big|_{y=0,a} &= 0 \\
\tau_{yz}^{1,2} \big|_{y=0,b} &= 0 \\
u_x^{1,2} \big|_{x=a} &= 0 \\
u_x^{1,2} \big|_{x=b} &= 0
\end{align*} \] (5)

The three elastic potentials for each plate satisfying the BC above have the form:

\[ \Phi = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \cos \frac{n\pi x}{a} \cos \frac{m\pi y}{b} \left[ A_{mn} \sin k_z z + B_{mn} \cos k_z z \right] \]

\[ \chi = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \cos \frac{n\pi x}{a} \cos \frac{m\pi y}{b} \left[ C_{mn} \sin k_z z + D_{mn} \cos k_z z \right] \]

\[ \psi = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} \left[ E_{mn} \sin k_z z + F_{mn} \cos k_z z \right], \] (6)

where \( k_z^2 = \alpha^2 - \left( \frac{n\pi}{a} \right)^2 - \left( \frac{m\pi}{b} \right)^2 \), \( k_z^2 = \beta^2 - \left( \frac{n\pi}{a} \right)^2 - \left( \frac{m\pi}{b} \right)^2 \).

The Fourier solutions in the \( x \) and \( y \) coordinates constitute a set of orthogonal functions on the ranges \( a \) and \( b \), respectively. From Eq. (6) each plate has six unknown Fourier constants. Since the two plates are totally bonded at their interface, complete continuity is enforced on the three displacements and the three stresses that are the components of the traction vector on that interface. These are stated as:

\[ \begin{align*}
\tau_{xx} \big|_{z=-h/2} &= \tau_{xx} \big|_{z=-h/2} \\
\tau_{yy} \big|_{z=-h/2} &= \tau_{yy} \big|_{z=-h/2} \\
\tau_{xy} \big|_{z=-h/2} &= \tau_{xy} \big|_{z=-h/2} \\
u_x \big|_{z=-h/2} &= \nu_x \big|_{z=-h/2} \\
u_y \big|_{z=-h/2} &= \nu_y \big|_{z=-h/2} \\
u_z \big|_{z=-h/2} &= \nu_z \big|_{z=-h/2}
\end{align*} \] (7)

For an incident plane acoustic pressure wave, the backward and forward scattered waves are given as:
\[ p_2(x, y, z) = p_{inc}(x, y, z) + p_i(x, y, z) + p_n(x, y, z) \]
\[ p_{inc}(x, y, z) = p_0 e^{i \omega (l_x x + m_y y + n_z z)} \]
\[ p_i(x, y, z) = p_0 e^{i \omega (l_x x + m_y y + n_z z) - i \alpha} \]
\[ p_n(x, y, z) = p_0 e^{-i \omega (l_x x + m_y y + n_z z)} \]
\[ p_i(x, y, z) = p_i(x, y, z) e^{-i \alpha} \]

where \( l_i = \sin \theta_0 \cos \phi_0 = l_x, m_i = \sin \theta_0 \sin \phi_0 = m_y, n_i = -\cos \theta_0 = -n_z \). The boundary conditions at the interface of plate #1 and acoustic medium #1 enforces continuity of the plate’s normal velocity to the normal acoustic particle velocity, the plate’s normal stress to the acoustic pressure, and the that the shearing stresses vanish at the interface of the plate and acoustic medium

\[ \tau_{xx} \big|_{z=-h/2} = \tau_{yy} \big|_{z=-h/2} = 0 \]
\[ \tau_{xx} \big|_{z=-h/2} = -p_i \big|_{z=0} \]
\[ u_{z1} \big|_{z=-h/2} = \begin{cases} -\frac{1}{\rho_i \omega^2} \frac{\partial p}{\partial z} |_{z=0} & 0 \leq x \leq a, \ 0 \leq y \leq b \\ 0 & \text{elsewhere,} \end{cases} \]

where \( p_i \) is the acoustic pressure on the surface of plate #1. Similarly, the boundary conditions at the interface of plate #2 and the acoustic medium #2 are expressible as

\[ \tau_{xx} \big|_{z=-h/2} = \tau_{yy} \big|_{z=-h/2} = 0 \]
\[ \tau_{xx} \big|_{z=-h/2} = -p_2 \big|_{z=0} \]
\[ u_{z2} \big|_{z=-h/2} = \begin{cases} \frac{1}{\rho_2 \omega^2} \frac{\partial p}{\partial z} |_{z=0} & 0 \leq x \leq a, \ 0 \leq y \leq b \\ 0 & \text{elsewhere,} \end{cases} \]

Substitution of the wave potentials (6) into the six interface BCs (7) yields six homogeneous algebraic equations on the twelve Fourier coefficients. Substitution of the wave potentials of Eq. (6) into the first two BC of Eqs. (9) and (10) satisfying vanishing shear stresses results in yet four more homogeneous equations on the twelve unknown Fourier coefficients.

3. SEMI-INFINITE ACOUSTIC MEDIA

Since the finite plate is in contact with semi-infinite acoustic media, one needs to perform two-dimensional Fourier transforms on the \( x \) and \( y \) coordinates. Applying the double Fourier transforms on the acoustic wave equations and the second and third boundary conditions of Eqs. (9) and (10) yields an expression for the BFS acoustic pressures in terms of the Fourier components of the respective plates’ normal displacements

\[ p_n^{(j)}(x, y, z) = -\frac{\partial^2 \partial t}{\partial x^2} \sum_j \sum_r \phi_{rj} \int_0^b \cos \frac{r \pi a}{a} \cos \frac{s \pi v}{b} e^{-i \frac{\pi v}{a} R - i \frac{\pi v}{b} R} dudv, \]

where \( \phi_{rj} \) are the Fourier coefficients of the plate normal surface displacements \( u_z \) and \( R = \sqrt{(x-u)^2 + (y-v)^2 + z^2} \). Substituting the elastic potentials in Eq. (6) into the formula for the normal stress \( \tau_{zz} \) and the expression for \( p_n^{(j)} \) from Eq. (11) into the second BC of Eqs. (9) or (10), and using the orthogonality of the eigenfunctions, results in the following equation for the normal stress components in terms of the Fourier component of the plate normal displacements:

\[ R_{nm}^{(j)} = -\frac{i \Omega_c \rho_c P_{ij}}{a} \sum_j \sum_r \phi_{rj} \phi_{rjm} + \lambda_j^2 \sigma_f n_m^{(2)} \]

where \( \lambda_j \) are the eigenvalues and \( \phi_{rj} \) are the eigenfunctions.
In Eq. (12) \( R_{\text{mm}}^{(j)} \) represents the Fourier component of the normal stress \( \tau_{\text{zz}} \), \( \sigma = b/a \) denotes the plate aspect ratio, and \( f_{\text{nm}}^{(2)} \) represents the Fourier component of the surface acoustic pressure

\[
f_{\text{nm}}^{(2)} = \frac{P_0}{2\pi} \left( \frac{b}{2a} \right)^2 \epsilon_n \epsilon_m \Omega^2 \Lambda^2 \left[ \frac{(-1)^n e^{i\Omega \lambda_j} - 1}{n^2 \pi^2 - \Omega^2 \Lambda^2} \right] \left[ \frac{(-1)^m e^{i\Omega \lambda_m} - 1}{m^2 \pi^2 - \Omega^2 \Lambda^2} \right].
\]

where \( \Omega_j = \omega a / c_j \), \( c_p = (\rho / \rho_j) (1 - \nu_j) \), \( \Lambda_j = c_{p_j} / c_j \). The expression \( Z_{\text{mm}}^{(i)} \) in Eq. (12) represents the mutual modal acoustic impedance of the \( n, m \) mode with the \( r, s \) mode defined in an integral form

\[
Z_{\text{mm}}^{(i)} = \frac{2i\sigma \Lambda_j \Omega_j}{\pi} \int_0^{1/\omega} F_m(u) G_m(v) e^{i\Omega \lambda_j} R_0 du dv,
\]

where \( R_0 = \sqrt{u^2 + v^2} \), and \( F_m \) and \( G_m \) are defined by

\[
F_m(u) = \int_0^{1/\omega} \cos \pi(u + u_1) \cos n\pi u_1 du_1,
G_m(v) = \int_0^{1/\omega} \cos \pi(v + v_1) \cos m\pi v_1 dv_1.
\]

These integrals were evaluated numerically, and it was shown by Sandman [9] that for the cross-coupled modes, \( Z_{\text{mm}}^{(i)} \) is negligible when \( r \neq n \) and \( s \neq m \). Thus, only the self impedance components \( Z_{\text{mm}}^{(i)} \) were included in this paper.

Finally, there results 12x12 system of non-homogeneous equations on the twelve unknown Fourier constants of the plate potentials. Solving these equations for the set of twelve constants, one can substitute these for the twelve constants for the \( mn^{th} \) component of the three displacements and six stresses at any point in the plates. These constants then can be substituted in Eq. (11) to evaluate the radiated acoustic pressures in media #1 and #2.

### 4. FARFIELD PRESSURE

The expression in Eq. (11) gives the radiated acoustic pressures in acoustic media #1 and #2. To obtain a formulation that is valid in the farfield, one can first convert the Cartesian coordinates to spherical ones, with an origin at (0, 0, 0). Performing approximations on the integrand, one obtains an expression for the farfield BFS acoustic pressures as:

\[
p_{\text{f}}^{(j)} = \frac{P_0}{\rho_0 a} e^{i\Omega \lambda_j} R_{\text{r}} = \frac{\sigma a}{2\pi} f(v_j) \frac{\rho_j}{\rho_i} \Omega_j^2 \Lambda_j \cos \theta \sin \phi \sin \frac{\Omega_j \Lambda_j \sin \theta \cos \phi}{\Omega_k} \sin \frac{\Omega_j \Lambda_j \sin \theta \sin \phi}{\Omega_k} \frac{\Omega_j^2 \Lambda_j}{\Omega_k^2 \Lambda_k} \sin^2 \theta \sin^2 \phi + \frac{\rho_j}{\rho_i} \Lambda_j^2 \Omega_j^2 \sin^2 \theta \sin^2 \phi \sum_{n=1,1}^{\infty} \sum_{m=1,1}^{\infty} Q_{\text{mm}}^j Q_{\text{kk}}^j \frac{1 - (-1)^n \exp(i\Omega \Lambda_j \sin \theta \cos \phi)}{n^2 \pi^2 - \Omega_j^2 \Lambda_j^2 \sin^2 \theta \cos^2 \phi} \frac{1 - (-1)^m \exp(i\Omega \Lambda_m \sin \theta \sin \phi)}{m^2 \pi^2 - \Omega_j^2 \Lambda_j^2 \sin^2 \theta \sin^2 \phi} + \frac{R}{\rho_0 a} \left[ p_{\text{f}}^{(2)} + p_{\text{m}}^{(2)} \right] e^{i\Omega \lambda_j / a}
\]

where \( \theta \) and \( \phi \) are the spherical Eulerian angles, and \( R \) is the radial distance to an observer. The farfield expression is split so that the contribution of the \( n = m = 0 \) mode is separated, it being the largest contributor to the farfield, and has a directivity function that is the same as that of a rigid piston i.e., two sinc functions.
5. NUMERICAL RESULTS AND DISCUSSION

First, to examine the role the thickness of the plate plays in the scattering from a monolithic plate, numerical results were obtained for a bilaminar plate made of two identical steel plates with $h_1/a = h_2/a = 0.005$ and $0.1$, resulting in a monolithic plate with $h/a = 0.01$ and $0.2$. These ratios span the range from a very thin plate to a fairly thick one. The damping factor used was $\eta = 0.001$ with the plate aspect ratio $b/a = 2$. Results are initially obtained for normal incidence only with water on both sides of the plate. For a thin plate, $h/a = 0.01$ the normalized scattering spectrum is shown vs. frequency in Fig. 2a. The spectra are almost identical for very high frequencies, indicating that the two plate surfaces were displaced equally, and hence the acoustic scattering from both surfaces of a thin plate is the same. The farfield directivities at $\Omega = 3$ (Fig. 2b) and at $\Omega = 10$ (Fig. 2c) show the diffraction caused by low order modes.

![FIGURE 2. Farfield response of a submerged monolithic steel plate with b/a=2 and h/a=0.01, (a) on-axis farfield acoustic pressure, (b) directivity at $\Omega = 3$, (c) directivity at $\Omega = 10$](image)

For a thick steel plate, $h/a = 0.1$, Fig. 3a shows the back- and forward-scattered pressures could be as much as 30 dB different and the spectra have maxima and minima corresponding to thickness symmetric and anti-symmetric resonances and anti-resonances of the thick plate. The directivities at anti-resonance frequency $\Omega = 17.4$ (Fig. 3b) and at resonance 34.8 (Fig. 3c) show the high number of modes contributing to those directivities.

Consider now a bilaminar plate made up of an elastomer bonded to steel, with the elastomer layer being three times the thickness of the steel such that $h/a = 0.1$, with the elastomer having a high damping coefficient of 0.3. The frequency spectra for a bilaminar plate with the incident wave impinging on the steel side (Fig. 4a) shows that there are no resonances or anti-resonances. This is primarily due to the high polymer damping. It also shows that the steel plate drives the much softer polymer to high response, resulting in a much higher forward scattering levels. The directivities at $\Omega = 3$, 10 (Figs. 4b, 4c) show almost an omni-directional or low frequency diffraction at that mode order. When the incident wave impinges on the polymer side (Fig. 5a), the soft polymer is driven to a high response resulting in a much higher backscattered pressure by as much as 110 dB. The directivities for this case show diffraction effects at a higher polymer mode order (Figs. 5b, 5c).

In conclusion, the thickness modes for thick plates have a very high influence on the BFS spectra and the directivities of monolithic plates. For a steel-elastomer bilaminar plate, the BFS pressures are highest on the elastomer side, and the spectra exhibit no thickness resonances and anti-resonances.
FIGURE 3. Farfield response of a submerged monolithic steel plate with $b/a=2$ and $h/a=0.1$, (a) on-axis farfield acoustic pressure, (b) directivity at $\Omega = 17.4$, (c) directivity at $\Omega = 34.8$.

FIGURE 4. Farfield response of a submerged bilaminar steel-elastomer plate with $b/a=2$, $h_1/a=0.075$ and $h_2/a=0.025$, excitation on steel surface, (a) on-axis farfield acoustic pressure, (b) directivity at $\Omega = 3$, (c) directivity at $\Omega = 10$. 

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FIGURE 5. Farfield response of a submerged bilaminar steel-elastomer plate with $b/a = 2$, $h_1/a = 0.025$ and $h_2/a = 0.075$, excitation on elastomer surface, (a) on-axis farfield acoustic pressure, (b) directivity at $\Omega = 3$, (c) directivity at $\Omega = 10$

ACKNOWLEDGEMENT

Work supported by NAVSEA Division Newport under the ONR/ASEE Summer Faculty Program.

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