4aSA7. The peculiarities of the non-axisymmetric frequency spectra of finite elastic cylinders
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A rigorous solution of three-dimensional boundary-value problem concerning the forced vibrations of a finite, elastic, isotropic cylinder is constructed analytically by means of the superposition method. With this solution, the resonances in non-propagating waves were investigated. Particularly, a survey of the frequency spectrum for an aluminum cylinder, vibrating with the circumferential order two, reveals the existence of a localized resonance, usually referred to as an end resonance, well below the cut-off frequency of the lowest real dispersion branch of an infinite cylinder. This phenomenon demonstrates the remarkable differences between the axisymmetric and non-axisymmetric end resonances of elastic cylinders. Comparison of the theoretical results with the experiments published elsewhere reveals an excellent agreement.

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INTRODUCTION

The problem of rigorous determination of resonant frequencies for structures in the shape of a cylinder, as well as accurate evaluation of their dependence on material properties and geometric parameters, has received a considerable importance in recent years in dynamic, nondestructive testing methods (Nieves et al., 2003, 2004; ASTM Standard, 2008).

First study concerning elastic vibrations within a circular infinite rod has been carried out for more than a hundred years ago (Pochhammer, 1876). Nevertheless, some peculiarities of resonant spectra for rods of finite dimensions continue to be an intriguing and not fully understood at the present time. A typical difficulty arises if one considers the axisymmetric wave \( L(0, 1) \) traveling in a cylinder of infinite length and then introduces a plane of termination; one then finds that the reflected waves are composed of the other modes \( L(0; 2, 3, \ldots) \) besides that of the incident wave. The possible wave numbers for reflected waves are the roots of Pochhammer’s dispersion equation. Some modes (real wave numbers) are found to propagate back into the rod; whereas others (complex wave numbers) form the infinite number of the standing waves that decay in amplitude with distance from the reflecting end. The resonance phenomenon in these standing waves are known in literature (see, for example (Gregory and Gladwell, 1989; Kari, 2003; Pagneux, 2012) or the early references in (Grinchenko and Meleshko, 1981)) as “end resonance”. There is one further complication. The end vibration represents an intricate coupling between the propagating mode and the system of evanescent waves. Thus if the strong interaction between these system occurs, the resonances in inhomogeneous waves may not be induced in a finite rod with a particular semi-length to radius ratio (\( B \) region in fig. 1a). Detailed studies (Meleshko, 1979; Holst and Vassiliev, 2000; Pagneux, 2012) revealed the existence of the uncoupled (pure) axisymmetric end resonances for the specific type of cylinders’ excitation and material properties, however, hardly achievable in practice, that is, Poisson’s ratio \( \nu = 0 \) (fig. 1b) and \( \nu \approx 0.127 \).

The purpose of this paper is to report the results for non-axisymmetric deformations of a cylinder, for which the pure end resonance exists for any value of Poisson’s ratio.
PREDICTIONS OF THE THEORY FOR ELASTIC WAVES IN CYLINDERS

The graph of fig. 2 displays the dispersion properties of non-axisymmetric steady-state waves \( u(r, \theta, z) \cdot e^{i\omega t} \equiv U(r) \cdot e^{i[\theta + k z + \omega t]} \) in an infinite cylinder with traction-free surface at \( r = a \). These properties are governed by the dispersion relation

\[
\mathcal{P} \cdot \mathcal{Q} - \mathcal{C} \cdot \mathcal{T} = 0, \tag{1}
\]

where

\[
\mathcal{P} = \frac{(\kappa^2 + q_s^2 + 2\ell^2)(\kappa^2 + q_s^2)}{4\kappa^2 \mathcal{J}_\ell(q_2)} - \frac{q_s^2 + \ell^2}{\mathcal{J}_\ell(q_2)} + \frac{q_s^2 + \ell^2}{2\kappa^2}, \quad \mathcal{Q} = \mathcal{J}_\ell(q_2) - \frac{q_s^2 + \ell^2}{2} - \frac{\ell^2}{2\mathcal{J}_\ell(q_1)},
\]

\[
\mathcal{C} = \mathcal{J}_\ell(q_2) - \frac{1}{2} - \frac{\kappa^2 + q_s^2 + 2\ell^2}{4\kappa^2 \mathcal{J}_\ell(q_1)} \ell, \quad \mathcal{T} = \left[ \frac{1}{\mathcal{J}_\ell(q_2)} - \frac{\kappa^2 + q_s^2 + 2\ell^2}{2\kappa^2 \mathcal{J}_\ell(q_1)} - \frac{\gamma_s^2}{2\kappa^2} \right] \ell,
\]

and

\[ q_s^2 + \gamma_s^2 = \kappa^2, \quad \gamma_s = \frac{\omega a}{c_s}, \quad \mathcal{J}_\ell(q_s) = q_s I'_\ell(q_s) \] (s = 1, 2).

Here \( c_1 \) and \( c_2 \) are the dilatational and shear wave velocities, respectively; \( I_\ell \) is the modified Bessel function of the order \( \ell \).

The curves \( F(2; 1, 2, \ldots) \) represent the dimensionless frequency \( \gamma_2 = \omega a/c_2 \) versus normalized propagation constant \( \kappa \) for circumferential waves of the order \( \ell = 2 \). The spectrum is plotted for Poisson’s ratio \( \nu = 0.344 \). The flexural modes \( F(\ell; 1, 2, \ldots) \) for other values \( \ell \geq 2 \) and \( \nu \) would be similar to that of fig. 2 (Meeker and Meitzler, 1964).

**Figure 2:** Non-axisymmetric dispersion spectrum of second circumferential order for infinite cylinder with Poisson’s ratio \( \nu = 0.344 \). Dimensionless propagation constant \( \kappa \) vs frequency parameter \( \gamma_2 = \omega a/c_2 \) represents flexural modes of propagation as dispersion branches \( F(2; 1, 2, \ldots) \). — : real or imaginary branches; - - - : complex branches. \( \gamma_{2,F} \) – cutoff frequency of second homogeneous mode \( F(2, 2) \); \( \gamma_{2,F}^* \) – minimum of real parts of first \( F(2, 1) \) and second \( F(2, 2) \) dispersion branches (critical frequency); \( \gamma_{2,E} \) – frequency of resonance in inhomogeneous waves \( F(2; 1, 2, \ldots) \) (end resonance).

Some of branches are shown as dashed (complex \( \kappa \)) and the others as solid lines (imaginary or real \( \kappa \)). The curves joined together by solid lines define the first three flexural modes traveling or decaying in the +z direction, from zero frequency on up. In the first mode \( F(2, 1) \) (dashed line) the propagation constant at low frequencies is complex. This branch of complex roots intersects the second branch of real roots \( F(2, 2) \) (solid line) at a minimum \( \gamma_{2,F}^* \) (critical
frequency). Higher in frequency, the $F(2,2)$ mode is cut off at the frequency $\gamma_{2,F} \approx 2.349$. Within a narrow frequency range below the critical frequency $\gamma_{2,F}^* < \gamma_{2,F}$ an infinite cylinder is cut off but the semi-infinite waveguide has the resonance $\gamma_{2,E} \approx 0.908 \cdot \gamma_{2,F}$ (Meleshko, 1980). Thus in what follows, attention will be drawn to determination of the conditions for which this resonance may exists in a finite cylinder.

**Boundary Problem and Summary of Superposition Method**

For the sake of completeness, we shall start from formulation of boundary problem. Therefore it is required to find the amplitudes of displacement vector $\mathbf{u}(r, \theta, z) \cdot e^{i\omega t}$ (where $r = r_1/a$, $\theta, z = z_1/a$ being the dimensionless cylindrical coordinates) which in the interior of the cylinder $0 \leq r \leq 1$, $0 \leq \theta \leq 2\pi$, $|z| \leq h = H/a$ satisfies the Lamè differential equation

$$c_1^2 \cdot \nabla^2 \mathbf{u} - c_2^2 \cdot \nabla \nabla \mathbf{u} + (\omega a)^2 \mathbf{u} = 0,$$

and which on the surface of the cylinder fulfills the antisymmetric (with respect to the middle plane $z = 0$) boundary conditions

$$\begin{align*}
\frac{\sigma_r}{2\mu} = \frac{\partial u_r}{\partial r} + \nu \text{div} \mathbf{u} = 0, & \quad \tau_{r\theta} = \frac{1}{\mu} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} = 0 & (r = 1, |z| \leq h); \\
\tau_{rz} = \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} = 0 & (r = 1, |z| \leq h) \quad \text{and} \quad (0 \leq r \leq 1, |z| = h); \\
\frac{\sigma_z}{2\mu} = \frac{\partial u_z}{\partial z} + \nu \text{div} \mathbf{u} = \pm e^{i\omega t} \sum_{j=1}^{\infty} \frac{J_{\ell}(\lambda_j r)}{J_{\ell}(\lambda_j)}, & \quad \tau_{z\theta} = \frac{1}{\mu} \frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z} = 0 & (0 \leq r \leq 1, |z| = h).
\end{align*}$$

Here $\mu$ is the shear modulus; $\lambda_j$ are the non-zero roots of the equation $J'_{\ell}(\lambda) = 0$ arranged in increasing order. To solve the boundary-value problem (2), (3) we employ the superposition method. The principal idea of the method consists of representing the displacement vector $\mathbf{u}(r, \theta, z)$ in the finite cylinder as the sum of two displacement vectors: one for an infinite cylinder with the radius $(a)$ equal to that of the original cylinder, expanded into the Fourier series over the interval $-h \leq z \leq h$; and the other for an infinite plate with thickness $(2h)$ equal to the finite cylinder length, expanded into the Dini and Fourier-Bessel series over the sector $0 \leq r \leq 1$. The resulting components of the antisymmetric displacements take the following form:

$$\begin{align*}
\mathbf{u}_r(r, \theta, z) &= \sum_{n=1}^{\infty} (-1)^{n+1} \left( x_n \left[ \frac{\kappa_n^2 + q_2^2}{2\kappa_n^2} I'_\ell(q_1) - I'_\ell(q_2) \right] + z_n \ell \right) \frac{I'_{\ell}(q_1 r)}{2I'_{\ell}(q_1)} \sin(\kappa_n z) + h \sum_{j=1}^{\infty} \left( \frac{p_2}{\lambda_j} \sinh(p_2 z) - \frac{\lambda_j^2 + p_2^2 \sinh(p_1 z)}{2\lambda_j p_1 \cosh(p_1 z)} \right) \frac{J'_{\ell}(\lambda_j r)}{J_{\ell}(\lambda_j)} \cdot e^{i\omega t}, \\
\mathbf{u}_\theta(r, \theta, z) &= \sum_{n=1}^{\infty} (-1)^{n+1} \left( x_n \left[ \frac{\kappa_n^2 + q_2^2}{2\kappa_n^2} I'_\ell(q_1) - I'_\ell(q_2) \right] + z_n \ell \right) \frac{I'_{\ell}(q_1 r)}{2I'_{\ell}(q_1)} \sin(\kappa_n z) + h \sum_{j=1}^{\infty} \left( \frac{p_2}{\lambda_j} \sinh(p_2 z) - \frac{\lambda_j^2 + p_2^2 \sinh(p_1 z)}{2\lambda_j p_1 \cosh(p_1 z)} \right) \frac{\ell J'_{\ell}(\lambda_j r)}{J_{\ell}(\lambda_j)} \cdot e^{i\omega t}, \\
\mathbf{u}_z(r, \theta, z) &= \sum_{n=1}^{\infty} (-1)^n \left( x_n \left[ \frac{2 q_2 I'_{\ell}(q_1 r)}{\kappa_n} - \frac{\kappa_n^2 + q_2^2 I'_{\ell}(q_1 r)}{2\kappa_n q_1 I'_{\ell}(q_1)} \right] + z_n \ell \right) \frac{I'_{\ell}(q_1 r)}{2I'_{\ell}(q_1)} \cos(\kappa_n z) + h \sum_{j=1}^{\infty} \left( \frac{p_2}{\lambda_j} \sinh(p_2 z) - \frac{\lambda_j^2 + p_2^2 \sinh(p_1 z)}{2\lambda_j p_1 \cosh(p_1 z)} \right) \frac{\ell J_{\ell}(\lambda_j r)}{J_{\ell}(\lambda_j)} \cdot e^{i\omega t},
\end{align*}$$

where $J_{\ell}$ is the ordinary Bessel function of the order $\ell$, and

$$\begin{align*}
p_s^2 + \gamma_s^2 &= \lambda_{s,j}^2, & q_s^2 + \gamma_s^2 &= \kappa_n^2 & (s = 1, 2); & \quad \kappa_n = \pi (n - \frac{1}{2})/h & (n = 1, 2, \ldots).
\end{align*}$$
Asymptotically the unknowns in (4) have the predefined structure (Grinchenko and Meleshko, 1981)

\[
\lim_{n \to \infty} x_n = \lim_{j \to \infty} y_j = G \equiv \text{const}, \quad \lim_{n \to \infty} z_n \cdot k_n^4 = \text{const}.
\]

(5)

As the consequence of the relation \([\sigma_r - \sigma_z](1, \theta, \pm h) = \pm 2\mu g(1)e^{i\ell\theta},\)
expressing the difference of the normal stresses at the cylinder’s rims, the following relation for the constant
\(G\) must valid:

\[
\pi \sum_{n=1}^{\infty} \left( x_n \left[ \mathcal{H}_n - \mathcal{P}_n \right] + z_n \left[ \mathcal{V}_n - \mathcal{E}_n \right] \right) + \pi \sum_{j=1}^{J} y_j \left[ \mathcal{R}_j - h \mathcal{W}_j \right] + 2G\chi h = \pi \sum_{j=1}^{J} g_j,
\]

(6)

where

\[
\begin{align*}
\mathcal{H}_n & = -\frac{q_n^2}{2\kappa_n}\mathcal{F}(q_n) - \left( \frac{\kappa_n^2 + \gamma_n^2}{2\kappa_n^2}\mathcal{F}(q_n) \right), \quad \mathcal{V}_n = \frac{\kappa_n^2 + \gamma_n^2}{2\kappa_n^2}, \quad \mathcal{E}_n = \mathcal{F}(q_n), \\
\mathcal{R}_j & = \frac{\lambda_j^2 + p_j^2}{2\lambda_j^2 p_1} \tanh(p_1h) - \frac{\lambda_j^2 - \ell^2}{\lambda_j^2} \tanh(p_2h)
\end{align*}
\]

and

\[
\gamma_0^2 = \frac{\gamma_1^2}{1-2\nu}, \quad \zeta = \frac{\gamma_2^2 - \gamma_1^2}{2}.
\]

The boundary conditions (3) for the shear stresses \(\tau_{rz}, \tau_{r\theta}\) were fulfilled identically on the corresponding boundaries by proper choice of the functions in the square brackets (4). The satisfaction of the remaining boundary conditions, taking the expressions (3) and (4) into account, for zero normal stress \(\sigma_r\) and non-selfequilibrating shear stress \(\tau_{r\theta}\) over the curved surface \((r = 1)\), along with the prescribed condition for the normal stress \(\sigma_z\) at the plane ends \((|z| = h)\) leads to three functional equations. In order to equate the left and right sides of these relations, we expand the left sides of the functional equations corresponding to the stresses \(\sigma_r, \tau_{r\theta}\) into the Fourier series over the interval \(-h \leq z \leq h\) and stress \(\sigma_z\) into the Dini series over the segment \(0 \leq r \leq 1\). As the result of these transformations, we obtain three infinite sets of linear algebraic equations for the unknowns \(x_n, y_j, z_n\). Subtracting and adding to the series, which contain the terms with \(x_n, y_j, z_n\), their asymptotic values (5), the infinite system is reduced to the following finite system:

\[
\begin{align*}
\sigma_r : \quad x_n \mathcal{P}_n + z_n \mathcal{E}_n + \sum_{j=1}^{J} y_j \mathcal{D}_j n + G \sum_{j=1}^{J} \mathcal{D}_j n = 0, & \quad n = 1, 2, \ldots, N; \\
\tau_{r\theta} : \quad x_n \mathcal{F}_n + z_n \mathcal{D}_n + \sum_{j=1}^{J} y_j \mathcal{G}_j n + G \sum_{j=1}^{J} \mathcal{G}_j n = 0, & \quad n = 1, 2, \ldots, N; \\
\sigma_z : \quad y_j \mathcal{R}_j + \sum_{n=1}^{N} \left[ (x_n - G) \mathcal{K}_n j + z_n \mathcal{L}_n j \right] + G \sum_{n=1}^{N} \mathcal{K}_n j = g_j, & \quad j = 1, 2, \ldots, J.
\end{align*}
\]

(7)

Here the following notations are used

\[
\begin{align*}
\mathcal{D}_j n & = \left[ \frac{\kappa_n^2 - \kappa_j^2}{\lambda_j^2 + q_j^2} \right] \left[ \frac{\lambda_j^2 - \ell^2}{2\lambda_j^2} - \frac{\gamma_j^2}{\lambda_j^2} \right], \quad \mathcal{G}_j n = \left[ \frac{\kappa_n^2 - \kappa_j^2}{\lambda_j^2 + q_j^2} \right] \left[ \frac{\lambda_j^2 - \ell^2}{2\lambda_j^2} + \frac{\gamma_j^2}{\lambda_j^2} \right] \frac{2\ell}{\lambda_j^2}, \\
\mathcal{K}_n j & = \left[ \frac{2\lambda_j^2}{\kappa_n^2 + p_1^2} - \frac{2\lambda_j^2}{\kappa_n^2 + p_2^2} + \frac{\gamma_j^2}{\kappa_n^2 + p_1^2} \right] \frac{\lambda_j^2}{\lambda_j^2 - \ell^2}, \quad \mathcal{L}_n j = \left[ \frac{\kappa_n^2 + \gamma_j^2}{\kappa_n^2 + p_1^2} \right] \frac{\ell \lambda_j^2}{\lambda_j^2 - \ell^2}, \\
\mathcal{R}_j & = p_2 h \coth(p_2 h) - \frac{\left( \lambda_j^2 + p_j^2 \right)^2}{4\lambda_j^2 p_1} h \coth(p_1 h).
\end{align*}
\]

The infinite sums in system (7) with terms \(\mathcal{D}_j n, \mathcal{G}_j n, \mathcal{K}_n j\) are not written out explicitly; they are long however straightforward.
FIGURE 3: (a) Non-axisymmetric resonant spectrum of circumferential order two represents frequency $\gamma_2$ vs length-to-radius ratio $h = H/a$ of finite aluminum rods. Resonances are antisymmetric with respect to middle plane $z = 0$. Plateau corresponds to end resonance frequencies $\gamma_{2,E}$. •: experimental observations of (McMahon, 1964); —: theoretical results. (b) Calculated antisymmetric displacements of curved surface at end resonance frequency $\gamma_{2,E}$.

**NUMERICAL RESULTS AND DISCUSSION**

The frequency spectrum $\omega a/c_2$ versus $H/a$ in fig. 3a was computed as follows. For given dimensions and Poisson’s ratio of the cylinder the determinant of the system (7) constitutes an implicit transcendental function of $h = H/a$ and $\gamma_2 = \omega a/c_2$. For a fixed value of $h$ this function is dependent on $\gamma_2$ alone. Thus the value of the function is evaluated at a prescribed start point $\bar{\gamma}_2$ and at interval of a specified $\Delta \gamma_2$ (typically $= 10^{-2}$) thereafter, up to and including a prescribed end point $\bar{\bar{\gamma}}_2$. A change of sign of the function across interval $\Delta \gamma_2$ indicates a root in that interval (resonant frequency). Then the iteration subroutine is applied to interval $(\bar{\gamma}_2, \bar{\bar{\gamma}}_2)$ until a root is obtained within a required accuracy. This procedure is repeated until a prescribed number of roots is calculated. The allowable relative error in resonant frequencies was $10^{-5}$ for the system with $N = [h] \cdot J = 30$.

The first spectral curve at the frequency spectrum (Poisson’s ratio $\nu = 0.344$ of aluminum) are shown in fig. 3a. The frequencies of short cylinders ($h < 2$) are in excellent agreement with experimental observations (McMahon, 1964) indicated by (•). This allows an evaluation of the validity of theoretical frequencies and therefore of the superposition method. As the spectral curve approaches the region $h = H/a \geq 3$, the resonant frequency $\gamma_2 = 0.908 \cdot \gamma_{2,E}$ becomes nearly independent of $h$, while the vibration patterns change to form with maximum motion occurring at the cylinder’s ends (fig. 3b).

Thus, if we suppose that a mechanism is available for suppling the desired (across the diameter) distribution of normal self-equilibrating stress, having a harmonic variation in time of a particular frequency $\omega = \gamma_{2,E} \cdot c_2/a$, to the ends of a cylinder with semi-length to radius ratio $H/a \geq 3$; we may induce the pure resonance in standing evanescent waves for any value of Poisson’s ratio. The experimental observations of these localized resonances, will be the subject of subsequent studies.
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