Dissipative effects in the response of an elastic medium to a localized force

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The effect of dissipation on the real part of the admittance of an elastic half-space is typically thought to be unimportant if the loss factor of the elastic medium is small. However, dissipation induces losses in the near field of the source and, provided the size of the source is small enough, this phenomenon can be more important than elastic wave radiation. Such losses give rise to a fundamental limit in the quality factor of an oscillator attached to a substrate. Near field losses associated with strains in the elastic substrate can actually be larger than intrinsic losses in the oscillator itself if the internal friction of the substrate is larger than the internal friction of the oscillator. This research was sponsored by the Office of Naval Research.

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INTRODUCTION

The average power absorbed by an elastic medium in response to a localized harmonic force \( F(\omega) \) is generally parametrized as

\[
\Pi(\omega) = \frac{1}{2} |F(\omega)|^2 R(\omega)
\]

where the resistive admittance \( R(\omega) \), often called the radiation admittance, is given by an appropriate spatial average of a Green’s function. For infinite structures, or large structures for which elastic waves excited by the source are scattered and then eventually dissipated, the absorption of energy has generally been interpreted as arising from the radiation of elastic waves into the medium; hence the term radiation admittance. Another potential source of energy absorption is dissipation associated with losses in the nearfield of the source, a mechanism first proposed by Photiadis (2011) for an elastic half-space. We may thus write

\[
R(\omega) = R_{rad}(\omega) + R_{inel}(\omega)
\]

where \( R_{rad}(\omega) \) corresponds to absorption associated with the radiation of elastic waves from the source, and \( R_{inel}(\omega) \) corresponds to inelastic absorption of elastic energy, either into excited carriers or thermal phonons. The quantity \( R_{inel} \) has typically been ignored (Wilson-Rae, 2008; Photiadis and Judge, 2004; Royston, 1999; Jimbo and Itao, 1968) in both the analysis of experimental results and in theoretical calculations. Thus far, there is no conclusive experimental evidence confirming this phenomena. Our purpose here is to describe energy absorption via inelastic absorption and to discuss the situations in which we expect this pathway to be important.

Generally, one would expect losses associated with dissipation in the near field to be small because the volume of the nearfield is small. However, as the source dimension \( a \) tends to zero, the local strain fields diverge and more than compensate for the small volume of the extreme near field of the source, distances of order \( a \). The net effect is in fact singular in three dimensions, \( R_{inel} \to \infty \) as \( a \to 0 \), and for sufficiently small source dimensions, inelastic absorption always dominates radiative absorption.

The remainder of the paper is organized as follows. In the following section, we discuss the simple example of an elastic half-space subjected to a localized normal stress. In this analysis, the singularity in the inelastic absorption is seen to result from very large wavenumbers of order \( a^{-1} \) and hence to be associated with distances of order the source size \( a \). The quantity \( R_{inel} \) is obtained analytically. We next examine the case of a general elastic structure, show that the qualitative result follows from general considerations, and argue that the analytical predictions are valid for any structure large compared to the source dimensions. The impact of our results on the measured \( Q \) of a normally attached oscillator, a geometry often employed in structural acoustics models (Strasburg and Feit, 1996; Soize, 1986) and which also often occurs in nanomechanical systems (Zalalutdinov et al., 2001; Hutchinson et al., 2004; Yang et al., 2002; Yasamura et al., 2000) is briefly considered. Finally, we give some closing remarks.

POWER ABSORPTION BY AN ELASTIC HALF-SPACE

The power absorbed by an elastic half-space in response to a localized normal stress \( \sigma \) over an area \( S \) of the surface is given by

\[
\Pi = \int_S dS \sigma(x,t)v(x,t) \approx \sigma(t) \int_S dS v(x,t)
\]

where \( v(x) \) is the normal velocity of the medium on the source area \( S \), and we have assumed that the stress \( \sigma \) is nearly constant. For assumed harmonic motion, \( \sigma(t) = \exp(-i\omega t)\sigma \),
\( v(x, t) = \exp(-i\omega t)v(x) = -i\omega \exp(-i\omega t)w(x) \), where \( w(x) \) is the normal displacement field in the medium, and the functions \( u(x), v(x), \sigma \) are complex and frequency dependent. We may write the time averaged, absorbed power in terms of the Green’s function \( G(x, x') \) of the half space to a normal \( \delta \)-function stress field as,

\[
\Pi = \frac{1}{2} Re \left( \sigma^* \int_S dS v(x) \right) = \frac{1}{2} \int_S dS' dS Re \left( -i\omega G(x, x') \right) = \frac{1}{2} |F|^2 R.
\]

Here the resistive admittance \( R \) is given explicitly by an average of the half-space Green’s function over the source area \( S \),

\[
R = \omega \int_S \frac{dS \, dS'}{S} \text{Im} G(x, x'),
\]

and \( F = \int_S dS \sigma \approx \sigma S \) is the total force applied to the half-space by the stress \( \sigma \). The analogy of the half-space to a one-dimensional oscillator with admittance \( A(\omega) = iX(\omega) + R(\omega) \) and velocity \( v(\omega) = A(\omega) F(\omega) \) is clear.

Using the formal solution for the response of an elastic half-space subjected to a localized normal force uniformly distributed over a disk of radius \( a \) given by Miller and Pursey (1954), Photiadis (2011) gives for the admittance,

\[
A(\omega) = \frac{2i\omega^3}{\pi a^2 \rho c_s^4} \int_0^\infty \frac{(k^2 - k_L^2)^{1/2}(J_1(ka))^2}{k\tilde{F}(k)} \, dk,
\]

\[
\tilde{F}(k) = (2k^2 - k_S^2)^2 - 4k^2(k^2 - k_L^2)^{1/2}(k^2 - k_S^2)^{1/2},
\]

with \( \rho \) the density of the medium, \( k_S = \omega/c_S(k_L = \omega/c_L) \) the shear(longitudinal) wavenumber, and \( J_1 \) the cylindrical Bessel function of order 1. The geometry is shown in Fig. 1 for the case in which the normal force is exerted by a simple oscillator, an example we shall examine later. The resistive admittance is given by \( R(\omega) = ReA(\omega) \).

The integrand in Eq. (6) has a simple pole at the Rayleigh wavenumber \( k_r \), and branch points at \( k = k_L \) and \( k = k_S \). The correct branch of the square roots is \((k^2 - k_S^2)^{1/2} \rightarrow -i(k_S^2 - k^2)^{1/2}\) for \( k < k_S \), and similarly, \((k^2 - k_L^2)^{1/2} \rightarrow -i(k_L^2 - k^2)^{1/2}\) for \( k < k_L \). If the dissipation of the medium is assumed to vanish, the real part of \( A(\omega) \) arises mostly from the pole contribution at \( k_r \), the excitation of Rayleigh waves, and to a lesser extent from wavenumbers smaller than \( k_S \) associated with the radiation of shear and longitudinal waves into the half-space. The inelastic
contribution \( R_{inel} \) vanishes. If on the other hand, we allow the dissipation \( \zeta \) of the elastic medium to be small but non-vanishing, we obtain a contribution from wavenumbers greater than the Rayleigh wavenumber corresponding to dissipation in the nearfield. For fixed \( \zeta \), the imaginary part of the integral in Eq. (6) corresponding to \( R(\omega) \) diverges as \( k \to \infty \) for \( a \to 0 \) and we therefore focus on the large wavenumber behavior of the integrand corresponding to the very nearfield.

Following Photiadis (2011), let the cutoff wavenumber \( k_\Lambda \) be far greater than \( k_r \) and consider the domain \( k > k_\Lambda \). In this domain, to leading order in \( k_r/k_\Lambda \), the integral simplifies to,

\[
R_{inel} \approx \frac{2\omega^2 c_L}{\pi \rho c_S^4} \frac{\alpha}{k_L^2 a^2} \int_\Lambda^\infty dx \frac{(J_1(xk_L))^2}{x^2} \tag{7}
\]

where

\[
\alpha = \frac{\zeta_S (\mu^2 - 2) + \zeta_L}{(\mu^2 - 1)^2}, \quad \mu = \frac{c_L}{c_S}. \tag{8}
\]

We have here allowed the shear and longitudinal waves to have different loss factors \( \zeta_S \) and \( \zeta_L \). Evidently, if we take the limit as \( a \to 0 \), the integral above is linearly divergent. Because of the linear divergence the integration is dominated by wavenumbers near the cutoff wavenumber \( a^{-1} \) and it is straightforward to obtain the most singular contribution as \( a \to 0 \) analytically (Photiadis, 2011),

\[
R_{inel} \approx \frac{8\omega^2 c_L}{3\pi^2 \rho c_S^4} \frac{\alpha}{k_L a}. \tag{9}
\]

For comparison, the radiative contribution is given by (Miller and Pursey, 1954)

\[
R_{rad} \approx \frac{\omega^2 c_L I}{2\pi \rho c_S^4} \tag{10}
\]

where \( I \) is typically less than or of the order of unity and must be evaluated numerically. The ratio of the inelastic to radiative contributions is, setting \( I = 1 \),

\[
\frac{R_{inel}}{R_{rad}} \approx \frac{16\alpha}{3\pi k_L a} \tag{11}
\]

and evidently if \( k_L a \leq \alpha \) inelastic losses exceed radiative losses. Generally, inelastic losses become more important in the domain of low frequency and small source dimensions.

### Inelastic Power Absorption in a General Structure: Universality

The extent to these phenomena generalize is important because we seldom encounter semi-infinite elastic media. For radiative losses, this is quite important because the total radiated power can be strongly influenced by interference. As such, in many experimental geometries, it is difficult to give an accurate prediction of these losses, and in most cases the result given in Eq. (10) should be taken as an order of magnitude estimate, perhaps to be averaged over frequency. Inelastic losses are quite different and indeed, we argue that the simple prediction given in Eq. (9) is independent of the geometry provided the structure is far larger than the source dimensions.

Since the wavenumbers that dominate the inelastic losses are of order \( a^{-1} \), the corresponding length scales from the source over which inelastic losses are important are of order \( a \). The dynamics underlying this result can be understood from the following qualitative argument which correctly yields the singular behavior in Eq. (9). The strain in the nearfield
resulting from the external force falls off as $r^{-2}$, where $r$ is the distance from the source. The associated stress thus also falls off as $r^{-2}$, and the total energy density in the nearfield behaves as $r^{-4}$. The total energy stored in the nearfield is thus of order

$$E_{nf} \propto \int d^3r \epsilon \cdot \sigma \propto \int_{r \geq a} d^3r r^{-4} \sim a^{-1},$$

and hence the rate of energy dissipation in the nearfield region is $\Gamma = \zeta E_{nf} \propto \zeta a^{-1}$ in agreement with Eq. (9).

Suppose now that just the area of the source can be regarded as planar. Then the argument leading to Eq. (5) giving the energy absorption of the system is still valid, albeit the Green’s function in now the appropriate Green’s function for the specific system under consideration rather than a half-space. In typical situations, the structure is composite, and composed of many elastic elements joined together in some fashion. In any case, assume the particular elastic member under excitation is large compared to the source dimensions.

The Green’s function appearing in Eq. (5) can be constructed in the following classical fashion. Suppose $G_0(x,x')$ is the Green’s function of a semi-infinite elastic medium subjected to the source $\sigma$. Then $G_0(x,x')$ satisfies the equations of elasticity everywhere in the half-space, has the appropriate singularity at the source, and obeys the correct boundary conditions in the neighborhood of the source. The correct Green’s function satisfies in addition appropriate boundary conditions which can be satisfied by adding homogeneous solutions of the equations of elasticity. These solutions are all regular, non-singular at the source location and thus do not contribute to the singular component of the field. This argument is rigorous in the case in which the actual structure is contained in the half-space, and we conjecture that the argument can be made entirely rigorous by extending the half-space Green’s function to satisfy the equations of elasticity theory in all space.

Thus in the computation of inelastic absorption, we may use the elastic half-space Green’s function, and Eq. (9) is valid for any system that is large compared to the source dimensions. Note that the system need not be large compared to a wavelength, and may indeed be small compared to a wavelength. This result is almost the opposite of the case for radiative losses, which can change by orders of magnitude as any dimension of the system becomes smaller than a wavelength and the system begins to behave as a plate (Photiadis and Judge, 2004).

**THE ROLE OF INELASTIC LOSSES ON THE $Q$ OF AN ATTACHED OSCILLATOR**

Consider a simple oscillator with an intrinsic loss factor $\zeta_0$, attached to an elastic structure on a disk of radius $a$ as shown in Fig. 1. Assuming no other sources of loss are present, the quality factor of the oscillator is $Q^{-1} = \zeta_0 + \zeta_r$ where $\zeta_r$ is the loss factor associated with absorption by the elastic half-space. Following Photiadis and Judge (2004), the loss factor associated with absorption by the elastic half space is given by

$$\zeta_r = \Pi / \omega E$$

where $\Pi$ is the average power transmitted by the oscillator into the structure, and $E$ is the average energy stored by the oscillator.

For harmonic motion with amplitude $x_0$, assuming that the motion of the attachment point is small compared to the motion of the mass, we have $\Pi = 1/2Re(A)|F|^2 = 1/2Re(A)(Kx_0)^2$, while $E = \langle T + U \rangle = 2 \langle U \rangle = 1/2Kx_0^2$, and hence

$$\zeta_r = M \omega_0 Re(A).$$
Using Eq. (9) for $Re(A)$, the total loss factor of a simple oscillator normally attached to an elastic structure is therefore

$$Q^{-1} \approx \zeta_0 + \frac{8c_L^2 a M \omega_0^2}{3\pi^2 \rho c_S^4 a},$$  

(15)

where we have assumed $a$ is small enough that near field dissipative losses are far greater than radiative losses. The above result appears to be singular as $a \to 0$, but because we have assumed the motion of the attachment point is small relative to the motion of the mass, Eq. (15) is modified in this limit.

Defining the frequency

$$\Omega = \sqrt{\frac{3\pi^2 \rho c_S^4 a}{8c_L^2 M}},$$  

(16)

Photiadis (2011) shows that Eq. (15) is modified to

$$Q^{-1} \approx \zeta_0 + \frac{a}{\omega_0^2 + \Omega^2}.$$  

(17)

if $a$ is small enough that $\Omega$ is smaller than or of order $\omega_0$. The frequency $\Omega$ is determined by the effective stiffness of the medium to the concentrated load and the mass $M$. Requiring a small displacement of the attachment point relative to that of the mass is equivalent to requiring $\Omega \gg \omega_0$, a stiff medium. As $a$ and therefore $\Omega$ decrease, $\zeta_r$ increases and approaches a maximum $\zeta_r = a$ when $\Omega \ll \omega_0$. In this domain, we find for the inverse $Q$ of the oscillator,

$$Q^{-1} \approx \zeta_0 + a.$$  

(18)

If the loss factor of the substrate is larger than that of the oscillator, near field losses can exceed internal losses.

**CONCLUDING REMARKS**

We have discussed inelastic absorption as a new mechanism for power absorption by an elastic structure subjected to an applied force. We have presented a brief derivation giving an analytical prediction of this result in the case of an elastic half-space. It was found there that the losses arise in the extreme nearfield of the source and as such are determined by the singular nature of the Green's functions of elastic media. As a consequence of this fact, we have predicted that the result is in fact universal, applying to any structure with dimensions far greater than the source dimensions. In closing, we have briefly commented on the impact of this phenomena on the $Q$ of an oscillator attached to a elastic structure.

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**REFERENCES**


