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4pSA7. Shock dynamics of random structures
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Predicting the response of a structure following an impact is of interest in situations where parts of a complex assembly may come into contact. Standard approaches are based on the knowledge of the impulse response function, requiring the knowledge of the modes and the natural frequencies of the structure. In real engineering structures the statistics of higher natural frequencies follows those of the Gaussian Orthogonal Ensemble, this allows the application of random point process theory to get a mean impulse response function by the knowledge of the modal density of the structure. An ensemble averaged time history for both the response and the impact force can be predicted. Experimental and numerical results are presented for beams in bending and longitudinal vibration.

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INTRODUCTION

Shock events are very likely to occur in many complex structures, including valves, relays heat exchangers, aeronautics and automotive assemblies. The prediction of the impact characteristics such as maximum impact force, impact duration and maximum acceleration is vital in order to estimate the possible damage resulting from a shock event. Predicting the impact response has attracted much interest since Timoshenko studied the impact of a bar. The problem of a beam impacting a stop is often investigated, because of the application to automatic reed type valves, switches in electrical relays, heat exchanger tubes and helicopter rotor blades. The classical approach is based on integral equations using the impulse response function (IRF). The IRF is usually written in terms of the modes and the natural frequencies of the structure. Langley presented an asymptotic approach and calculated the impact characteristics assuming an impact force to be a half-period sine. The key aspect of the asymptotic method is that the statistics of higher natural frequencies of nearly all uncertain systems follow those of the Gaussian Orthogonal Ensemble (GOE). In this paper, a new method is presented to predict the response of a structure to an impact, combining the convolution approach with the asymptotic method, based on knowledge of a mean impulse response function. The response can be well predicted as long as reflected waves from the boundaries do not affect the response: usually impacts have a short duration and then the method can be a useful and fast prediction tool. The theory is applied to beams in bending and longitudinal vibrations and results are compared with numerical, experimental and published work.

THEORY

The response of a structure to a generic force \( F(t) \) in the time domain can be expressed using the convolution between the Impulse Response Function \( h(t) \) and the force:

\[
    x(t) = x_0(t) + h(t) * F(t)
\]

where \( x_0(t) \) is an assigned motion that would be followed in the case of no impact. It can be used to assign the initial condition for the impact and also to take into account an eventual rigid body motion following the impact. For an impact with a point contact, the force can be expressed in terms of a nonlinear spring according to the Hertzian theory:

\[
    F(t) = K\delta^{3/2}
\]

where \( \delta \) is the local deformation and \( K \) is the contact stiffness. Since the force depends on the displacement \( x(t) \), Eq. (1) can be solved step by step at discrete time intervals. The Impulse Response Function at the driving point is available in closed form for some particular cases but in general it can be expressed with a modal expansion. Neglecting damping the result is:

\[
    h(t) = \sum_n \bar{\phi}_n \alpha^{-3/2} \sin(\omega_n t)
\]

where \( \bar{\phi}_n \) is the \( n \)th mass-normalised mode calculated at the driving point and \( \omega_n \) is the corresponding natural frequency. The standard deterministic approach uses knowledge of the modes of the structure in order to build the IRF as expressed in Eq. (4). As mentioned earlier, during an impact the response is likely to be dominated by the higher order modes whose sensitivity to uncertainties is very high. Their natural frequencies can be considered as randomly distributed along the frequency axis. This allows the application of random point process theory, leading to a mean IRF over the ensemble given by:

\[
    \mathbb{E}[h(t)] = M^{-1} \int \nu(\omega) \alpha^{-3} \sin(\omega x) \, d\omega
\]
where $V(\omega)$ is the modal density of the structure, i.e. the average number of resonance modes in a unit frequency band, and since the mode shapes are scaled to unit mass the result that $\mathbb{E}[\phi_n^2] = M^{-1}$ has been used, where $M$ is the total mass of the structure. Expressions for the modal density are available for a range of simple structures, such as beams, plates and thin cylinders. $\mathbb{E}[h(t)]$ has a two-fold interpretation: it can be seen as a mean impulse response over an ensemble of random structures with random natural frequencies due to small uncertainties or different boundary conditions, or as the impulse response of an infinite structure. In a more general context Shorter and Langley\textsuperscript{12} showed that the response of a random structure can be seen as the superposition of a direct field, described by the outgoing waves generated at the driving point, and a reverberant field due to the scattering of the waves at the random boundary. According to this interpretation, the response calculated with the method presented in this paper relates to the direct field due to the impact force. Considering the infinite-structure interpretation, it is easy to verify that the result given by Eq. (5) coincides with the closed form solutions available, for example, for an infinite beam and an infinite plate under flexural vibration\textsuperscript{11}.

### Beam in Flexural and Longitudinal Vibration

For the case of a beam in bending vibration, the modal density is given by\textsuperscript{13}

$$V = V_0 \omega^{1/2} = \frac{L}{2\pi E\! I} \left(\frac{m}{E I}\right)^{1/4} \omega^{1/2} \tag{5}$$

where $L$ is the length of the beam, $m$ is the mass per unit length, $E$ is the Young’s modulus and $I$ is the second moment of area of the cross section. In this case the integration of Eq. (4) has a closed form solution given by

$$\mathbb{E}[h(t)] = M^{-1} V_0 (2\pi t)^{1/2} \tag{6}$$

Eq. (6) does not depend on the length and agrees with the analytical result found for an infinite beam.

The modal density for the longitudinal vibrations of a rod is constant and is given by\textsuperscript{13}

$$V = V_0 = \frac{L}{\pi} \left(\frac{\rho}{E}\right)^{1/2} \tag{7}$$

The mean impulse response in this case is given by:

$$\mathbb{E}[h(t)] = \frac{\pi}{2} M^{-1} V_0 \text{sign}(t) \tag{8}$$

### RESULTS

**Impact of a Cantilever Beam with a Hard Stop**

Numerical and experimental verifications of the impact theory are now performed. A first numerical test is done on a cantilever beam with the following dimensions, length $L = 547$ mm, width 25 mm, thickness 6 mm. The beam is made of steel with density $\rho = 7800$ kg/m$^3$, Young’s modulus $E = 210$ GPa. The impact force was calculated for the case of a vibrating beam impacting a linear spring at three different locations and it was shown that the responses were in agreement as long as the reflections from the boundaries do not affect the response. In Figure 1 the impact force calculated using the asymptotic impulse response function presented in the previous section shows very good agreement with the first part of the deterministic solution, confirming the ability to predict the response for an average or infinite structure, depending on the preferred interpretation.
Ball Impacting a Free-Free Beam

The theory has also experimentally verified for a rigid ball hitting the same beam from a height of 23 mm. The motion of a steel ball of mass $M$ falling from a distance $H$ above the structure is governed by

$$M\ddot{y}(t) = F(t) - Mg$$

where $g$ is the gravitational acceleration and $y(t)$ is the ball coordinate, positive upwards and with the origin at the starting position. The equation is solved by integrating according to the Euler scheme given by

$$F(t_k) = K(x(t_{k+1}) - y(t_{k+1}) - H)_{3/2}$$

$$\dot{y}(t_k) = \frac{F(t_k)}{M} - g$$

$$y(t_k) = y(t_{k-1}) + 2y(t_{k-1}) - y(t_{k-2})$$

Eqs. (10) are solved together with the convolution that describes the dynamics of the flexible structure. The simulated displacement of the beam at the driving point and of the ball are shown in Figure 2. The ball bounces back after the relative displacement reaches a maximum. The acceleration is shown in Figure 3 together with the experimental result: the matching for the first part is very good, until reflection from the boundaries affects the result. The results are also compared with the acceleration calculated using the impulse response function as a modal expansion for a free-free beam: in this case the simulation is able to catch also the reflection from the boundaries and it looks closer to the experimental result; 94 modes were included in the simulation. It should be mentioned that the match in this case is extremely good since all the parameters were known with little uncertainty. In a real situation where a beam structure is part of a complex assembly, the modal properties would only be known approximately and the method proposed here would be the only feasible way to estimate the impact characteristics. The contact force is shown in Figure 4 and it is well approximated by a $1/2$ period sine shape.
Impact of a Ball with a Rod

In this section the results published by Hu et al.\textsuperscript{14} are reproduced for a further benchmarking. Hu et al.\textsuperscript{14} presented numerical and experimental work to predict the response of a rod impacted by a sphere. The impact happens at the edges of the structures: for this reason the asymptotic IRF has to be multiplied by a concentration factor equal to two. The ball velocity for the impact tested in ref \textsuperscript{14} is shown in Fig. 5(a), and is in very good agreement with the published result. The strain at the impact point is shown in Fig. 5(b) and the amplitude is consistent with the result shown in Fig. 5 of ref. \textsuperscript{14} where the strain at 3 cm from the impact was shown, the impact velocity is 0.3084 m/s.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{fig2}
\caption{Displacement for the ball/beam impact.}
\end{figure}

\begin{figure}
\centering
\includegraphics[width=\textwidth]{fig3}
\caption{Acceleration for the ball/beam impact.}
\end{figure}
CONCLUSIONS

A new method has been proposed to predict the response of a structure to an impact with a rigid or flexible body. The contact force was expressed with a Hertzian spring contact and the convolution between the Impulse Response Function and the force is used to predict the response. Using an assumption of randomness of the structure, only the modal density and the mass of the structure are needed in order to derive the Impulse Response. The theory has been numerically and experimentally verified for a beam and road showing excellent agreement. The impact theory presented in this work offers a very fast prediction tool for a large spectrum of situations.
REFERENCES