Dynamic analysis of annular sector plate with general boundary supports

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Dynamic behavior of annular sector plate is an important research topic since they have been extensively used in practical engineering applications. However, the dynamic analysis of annular sector plates with general boundary supports is rarely studied in literature. In this investigation, an analytical method is presented for the vibration analysis of annular sector plates with general elastic boundary supports. Unlike most existing framework, arbitrary elastic boundary supports can be easily realized by setting the stiffness of the two types restraining springs. The displacement field is universally expressed as a new form of trigonometric series expansions with a drastically improved convergence as compared with the conventional Fourier series. Mathematically, such a double Fourier series is capable of representing any function (including the exact displacement solution) whose third-order partial derivatives are continuous over the area of the plate. Thus, the double Fourier series solution to the dynamic analysis of the structure is obtained by employing the Raleigh-Ritz method. The accuracy and reliability of the current method are validated by both FEA and reference results under various boundary conditions. The present method can be directly applied to other more complicated boundary conditions and other shape plates.

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INTRODUCTION

Annular sector plates are a type of structural components commonly used in practical applications. The vibrational characteristics of sector plates are thus of great interest to engineers and designers. Although there is a vast pool of studies about vibrations of circular and rectangular plates [1], relatively few results are reported for annular sector plates.

In particular, a general technique was developed by Leissa [1] to obtain exact modal frequencies for plates which are simply supported along the radial edges, and have arbitrary boundary conditions at the circular edges. This method utilizes the well-known Bessel function solutions for a circular plate by allowing the functions to have non-integer orders. His following work [2] using Ritz method advocated that use of the ordinary Bessel functions solution is incorrect for sectorial thin plates having simply supported radial edges and sector angle \( \alpha > 180^\circ \). Liew et al. [3] reviewed many investigations about the vibration of thick plates published before 1993. It is shown that a majority of them is focused on the classical boundary conditions (simply supported, clamped or free edges). In comparison, other more complicated boundary conditions such as elastic boundary supports are rarely attempted. A closed-form solution is proposed by Kim and Yoo [4] in which the displacements is expressed in terms of trigonometric and exponential functions under the polar coordinate system. Malekzadeh[5] et al employed a three dimensional hybrid method to study the dynamic response of thick annular sector plates with simply supported radial edges. Ramakrishnan and Kunukkasseril [6] solved the vibration problem of an annular sector plate with simply supported radial edges and arbitrary conditions along the circumferential edges. Aghdam [7] presented an approximate solution for bending deformation of thin sector plates using extended Kantorovich method in which the fourth-order governing equation is converted into two ordinary differential equations. Employing the \( pb-2 \) Rayleigh-Ritz method, Xiang [8] tackled the vibration problem of annular sector Mindlin plates. Frequency parameters for annular sector plates with different geometry parameters and boundary conditions were presented. Wang extended the differential quadrature (DQ) method to analyze the free vibration of thin sector plates.

Previously, an Improved Fourier Series Method (IFSM) was proposed by Li [9] for the vibration of an arbitrarily supported beam. This analytical solution method has been subsequently extended to other structural components such as beams [10], plates [11], shells [12], and built-up structures [13]. The objective of this study is to realize and extend the IFSM to the dynamic analysis of annular sector plates under various boundary conditions, including the general elastic restraints. The displacement solution is expressed as a new form of trigonometric expansion with accelerated convergence, and the expansion coefficients are solved using the Rayleigh–Ritz technique. Several numerical examples are carried out to investigate the dynamic characteristics of annular sector plate.

MODEL DEVELOPMENT

An annular sector plate (consisted with two radial and two circular edges) and the coordinate system used in this investigation are shown in Fig. 1. This plate is of constant thickness \( h \), inner radius \( a \), outer radius \( b \), width \( R \) of plate in radial direction and sector \( \alpha \). The plate geometry and dimensions are defined in a cylindrical coordinate system \((r, \theta, z)\). The boundary conditions for the bending motion can be generally specified in terms of two kinds of restraining springs (translational and rotational) along each edge, resulting in four sets of distributed springs of arbitrary stiffness values. The classical boundary conditions can be easily achieved by setting the stiffness coefficients into zero or infinitely large number. The units of the linear displacements and rotational spring stiffness for vibration components are N/m and Nm/rad.

In the previous papers [11], each displacement component of a rectangular plate is expressed as a 2-D Fourier cosine series supplemented by eight auxiliary terms which are introduced to accelerate the convergence of the series expansion. In this study a similar, but much simpler and more concise, form of series expansion is employed to expand the displacement of an annular sector plate.

\[
    w(r, \theta) = \sum_{m,n=0}^{\infty} A_{mn} \Psi_m(r) \Psi_n(\theta) (\quad r = r + a) \tag{1}
\]

where \( A_{mn} \) denotes the expansion coefficients, and

\[
    \Psi_m(r) = \begin{cases} 
    \cos \lambda_m r & m \geq 0 \\
    \sin \lambda_m r & m < 0
    \end{cases} \quad \lambda_m = m\pi / R
\]

where \( m \geq 0 \) is a non-negative integer, \( n \) is a non-negative integer, and \( \lambda_m \) is the eigenvalue.

The basis function \( \Psi_n(\theta) \) in the \( \theta \) direction is also given by Eq. (2) except for that \( \lambda_n = n\pi / \alpha \).
The sine terms in the above equation are introduced to overcome the potential discontinuities, along the edges of the plate, of the displacement function when it is periodically extended and sought in the form of trigonometric series expansion. As a result, the Gibbs effect can be eliminated and the convergence of the series expansion can be substantially improved.

The plate equation demands that the third-order derivatives are continuous and the fourth-order derivatives exist everywhere over the surface area of the plate. Because the smoothness (or, explicitly, the convergence rate) of the current series expansion can be managed, at will, over the solution domain, the unknown Fourier coefficients can be obtained from either a weak or strong formulation. In seeking for a strong form of solution, the series is required to simultaneously satisfy the governing equation and the boundary conditions exactly on a point-wise basis. As a consequence, the expansion coefficients are not totally independent; the negatively-indexed coefficients are related to the others via the boundary conditions. In a weak formulation such as the Rayleigh-Ritz technique, however, all the expansion coefficients are considered as the generalized coordinates independent from each other. The strong and weak solutions are mathematically equivalent if they are constructed with the same degree of smoothness over the solution domain. The Rayleigh-Ritz technique will be adopted in this study since the solution can be obtained much easily. More importantly, such a solution process is better suitable for future modeling of built-up structures.

The Lagrangian for the annular sector plate can be generally expressed as

$$L = V_p + V_s - T - W_{ext}$$

(3)

where $V_p$ denotes the strain energy of the plate; $V_s$ designates the potential energy stored in the boundary springs; $T$ represents the kinetic energies corresponding to the vibration of the annular sector plate; and $W_{ext}$ is the work done by the external force.

For a purely bending plate, the strain energy can be expressed as

$$V_p = \frac{D}{2} \int_0^a \int_0^b \left[ \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} \right]^2 - 2(1 - \mu) \left( \frac{\partial w}{\partial r} \right)^2 \left( \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} \right) \right] r dr d\theta$$

(4)

where $D = Eh^3/(12(1-\mu^2))$ is the bending rigidity of the plate; $\mu$ is the Poisson’s ratio of the plate material.

The potential energies stored in the boundary springs are calculated as

$$V_s = \frac{1}{2} \int_0^a \left[ a \left[ k_a w^2 + K_a \left( \frac{\partial w}{\partial r} \right)^2 \right]_{\theta=\alpha} + b \left[ k_b w^2 + K_b \left( \frac{\partial w}{\partial r} \right)^2 \right]_{\theta=\alpha} \right] d\theta$$

$$+ \frac{1}{2} \int_\alpha^\beta \left[ \left[ k_a w^2 + K_a \left( \frac{\partial w}{\partial r} \right)^2 \right]_{\theta=\alpha} + \left[ k_b w^2 + K_b \left( \frac{\partial w}{\partial r} \right)^2 \right]_{\theta=\beta} \right] dr$$

(5)

where $k_a$ and $k_b$ ($k_0$ and $k_a$) are linear spring constants, and $K_a$ and $K_b$ ($K_0$ and $K_a$) are the rotational spring constants at $r=a$ and $b (\theta=0$ and $\alpha)$, respectively.

By neglecting rotary inertia, the kinetic energy of the annular sector plate is given by
\[
T = \frac{1}{2} \rho \omega^2 \int_a^b \int_0^{2\pi} w^r \, dr \, d\theta 
\]
where \( \rho \) is plate mass density, \( \omega \) is frequency in radian.

The work done by external force can be written as
\[
W_{ext} = \int_a^b \int_0^{2\pi} f(\tau, \theta) w(\tau, \theta) (\tau + a) \, d\tau \, d\theta 
\]
where \( f(\tau, \theta) \) is the external force function acted on the annular sector plate.

For a point force, the force function can be expressed as
\[
f(\tau, \theta) = F \delta(\tau - \tau_c) \delta(\theta - \theta_c)
\]
where \( F \) is the amplitude of the point force, \( \delta(\cdot) \) is Dirac delta function and \( (\tau_c, \theta_c) \) is the position where the point force is applied.

Substituting Eqs. (4-7) into the Lagrangian function Eq. (3) and minimizing Lagrangian against all the unknown Fourier coefficients, one can obtain a system of linear equations in a matrix form as
\[
(K - \omega^2 M) A = F 
\]
where \( A \) is a vector which contains all the unknown Fourier expansion coefficients, and \( K \) and \( M \) are the stiffness and mass matrices, respectively. \( F \) is the load vector. It can be described as
\[
F = F \Psi(w) \Psi(\theta)
\]

Eq. (9) represents a standard matrix characteristic equation from which all the eigenpairs can be determined by solving a standard matrix eigenvalue problem. Once the generalized coordinates, \( A \), is determined, the corresponding mode shape or displacement field can be constructed by substituting \( A \) into Eq. (1). For a given excitation frequency \( \omega \), the response vector \( A \) can be directly obtained as
\[
A = (K - \omega^2 M)^{-1} F 
\]

To overcome the numerical instability and singularity encountered at modal resonances, the damping of the structure is introduced in the form of complex Young’s modulus.
\[
\tilde{E} = E(1 + j\eta)
\]
where \( \eta \) is the damping coefficient.

After the Fourier expansion coefficient vector \( A \) and the displacement are determined, the mobility at any position on the annular sector plate can be calculated from
\[
Y = \frac{j\omega w}{F} 
\]

Since the displacement is constructed with the same smoothness as required of a strong form of solution, other variables of interest such as shear forces and power flows can be calculated directly, and perhaps more accurately, by applying appropriate mathematical operations to the displacement function.

**NUMERICAL RESULTS AND DISCUSSION**

**Modal Analysis**

To demonstrate the accuracy and usefulness of the proposed technique, several numerical examples will be presented firstly. Then numerical analyses are carried out to explore the dynamic characteristics of an annular sector plate with general boundary conditions. The material properties and structure dimensions of the annular sector plate are taken as follows. The plate is identically made of steel with density \( \rho = 7850 \text{ kg/m}^3 \), Young’s modulus \( E = 2.07 \times 10^{11} \text{ Pa} \), damping coefficient \( \eta = 0.01 \), Poisson’s ratio \( \mu = 0.3 \) and thickness \( h = 0.005 \text{ m} \). The inner radius, outer radius and sector angle are 0.4m, 1m and \( \frac{\pi}{3} \), respectively.

First, consider a completely clamped annular sector plate. A clamped B.C. can be viewed as a special case when the stiffness constants for both sets of restraining springs become infinitely large (represented by a very large number, \( 5.0 \times 10^{13} \), in the numerical calculations). In identifying the boundary conditions, letters C, S and F have been used to indicate the clamped, simply supported, and free boundary condition along an edge, respectively. The boundary conditions for a plate are fully specified by using four letters with the first one indicating the B.C. along the first edge, \( r = a \). The remaining (the second to the fourth) edges are ordered in the counterclockwise direction. The first six non-dimensional frequency parameters, \( \Omega = \omega b^2 \sqrt{\rho h/D} \), are tabulated in Table 1 together with an FEM prediction. To examine the convergence and numerical stability of the current technique, several sets of results.
are simultaneously presented obtained with different numbers of the expansion terms, $M=N=5$, 6, 7…, 12. An excellent convergence behavior is observed. For simplicity, the setting, $M=N=12$, will be used in the subsequent calculations.

**TABLE 1.** Frequency parameters $\Omega = \omega b^2 \sqrt{\rho h/\bar{D}}$, for completely clamped annular sector plate

<table>
<thead>
<tr>
<th>Truncate number</th>
<th>1</th>
<th>2</th>
<th>Mode sequence number</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
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<tr>
<td>$M=N=5$</td>
<td>65.709</td>
<td>78.421</td>
<td>101.35</td>
<td>133.81</td>
<td>175.31</td>
<td>176.82</td>
<td></td>
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<tr>
<td>$M=N=6$</td>
<td>65.701</td>
<td>78.415</td>
<td>101.32</td>
<td>133.80</td>
<td>173.86</td>
<td>175.72</td>
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<td>$M=N=7$</td>
<td>65.700</td>
<td>78.400</td>
<td>101.31</td>
<td>133.75</td>
<td>173.84</td>
<td>175.70</td>
<td></td>
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<tr>
<td>$M=N=8$</td>
<td>65.699</td>
<td>78.399</td>
<td>101.31</td>
<td>133.75</td>
<td>173.83</td>
<td>175.70</td>
<td></td>
</tr>
<tr>
<td>$M=N=9$</td>
<td>65.699</td>
<td>78.395</td>
<td>101.30</td>
<td>133.74</td>
<td>173.83</td>
<td>175.69</td>
<td></td>
</tr>
<tr>
<td>$M=N=10$</td>
<td>65.698</td>
<td>78.395</td>
<td>101.30</td>
<td>133.74</td>
<td>173.83</td>
<td>175.69</td>
<td></td>
</tr>
<tr>
<td>$M=N=11$</td>
<td>65.698</td>
<td>78.394</td>
<td>101.30</td>
<td>133.74</td>
<td>173.82</td>
<td>175.69</td>
<td></td>
</tr>
<tr>
<td>$M=N=12$</td>
<td>65.698</td>
<td>78.394</td>
<td>101.30</td>
<td>133.74</td>
<td>173.82</td>
<td>175.69</td>
<td></td>
</tr>
<tr>
<td>FEM</td>
<td>65.710</td>
<td>78.385</td>
<td>101.28</td>
<td>133.72</td>
<td>173.85</td>
<td>175.84</td>
<td></td>
</tr>
</tbody>
</table>

Next, an annular sector plate with combination of classical boundary conditions (CSCF) is investigated. The simply supported condition is simply produced by setting the stiffnesses of the translational and rotational springs to $\infty$ and 0, respectively; and the free edge condition by setting both stiffnesses to zero. The first six non-dimensional frequency parameters are shown in Table 2 together with the results calculated using an ABAQUS model. A good agreement is observed between the current and the FEM solutions.

**TABLE 2.** Frequency parameters, $\Omega = \omega b^2 \sqrt{\rho h/\bar{D}}$, for annular sector plate (CSCF)

<table>
<thead>
<tr>
<th>Mode sequence number</th>
</tr>
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<tr>
<td>The current method</td>
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<tr>
<td>FEM (ABAQUS)</td>
</tr>
</tbody>
</table>

Finally, consider a simply supported annular sector plate, but with rotational restraint, along each edge. The first six frequency parameters are presented in Table 3. Due to a lack of analytical solutions, the numerical results calculated using an FEM (ABAQUS) model are given there for comparison. It can be seen that a good agreement is observed between the current solution and the FEM results. Plotted in Fig. 2 are the first four mode shapes for the annular sector plate with rotational restraint $K=10^4$ Nm/rad.

**TABLE 3.** Frequency parameters, $\Omega = \omega b^2 \sqrt{\rho h/\bar{D}}$, for simply supported annular sector plate with uniform rotational restraint along each edge

<table>
<thead>
<tr>
<th>$K$(Nm/rad)</th>
<th>1</th>
<th>2</th>
<th>Mode sequence number</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
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<tbody>
<tr>
<td>1</td>
<td>32.769</td>
<td>46.448</td>
<td>68.929</td>
<td>99.012</td>
<td>115.96</td>
<td>131.79</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(32.764$^a$)</td>
<td>(46.438)</td>
<td>(68.915)</td>
<td>(99.002)</td>
<td>(115.97)</td>
<td>(131.79)</td>
<td></td>
</tr>
<tr>
<td>$10^4$</td>
<td>64.909</td>
<td>77.566</td>
<td>100.39</td>
<td>132.67</td>
<td>172.43</td>
<td>173.71</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(64.913$^a$)</td>
<td>(77.566)</td>
<td>(100.39)</td>
<td>(132.67)</td>
<td>(172.43)</td>
<td>(173.71)</td>
<td></td>
</tr>
<tr>
<td>$10^5$</td>
<td>65.699</td>
<td>78.394</td>
<td>101.30</td>
<td>133.74</td>
<td>173.84</td>
<td>175.69</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(65.706$^a$)</td>
<td>(78.398)</td>
<td>(101.31)</td>
<td>(133.75)</td>
<td>(173.86)</td>
<td>(175.74)</td>
<td></td>
</tr>
</tbody>
</table>

$^a$Results in parentheses are calculated using ABAQUS.
Based on the mobility theory, the vibration response of annular sector plates will be studied in this section. Firstly, to evaluate the prediction accuracy of the current method, two velocity mobility curves obtained from this method are compared with those of FEM solutions under the clamped boundary conditions. The geometry and material parameters are kept the same as mentioned above. A normal point force of unit amplitude is now applied at \((\pi/3, (b-a)/2)\) on the plate surface. The response point location for calculation of transfer mobility is \((\pi/2, 2(b-a)/3)\). These two comparison curves are plotted in Fig. 3(a) and 3(b). The truncated numbers of terms \(M=N=12\) are also used in this calculation. It can be found that the current method can predict the structure response with excellent accuracy.

Now, let us use this model to study the dynamic characteristics of annular sector plates under various boundary conditions. Fig. 4 shows the driving point and transfer point mobility for simply supported annular sector plate. It can be seen that the number of peaks for this case is more than clamped plates. As expected, the resonant peak occurred at its corresponding modal frequency. Therefore, all the factors, which can affect the modal characteristics, have impact on the structure mobility. It also should be noticed that changing boundary condition of annular sector plate will just cause the variation of the system matrix \(K\), which further interacts with the system mass matrix \(M\) and determines the plate modal characteristics as addressed in the above section. To understand the corresponding effects on the responses, we now consider an annular sector plate with uniform edge spring stiffness along each edge. The translational and rotational spring stiffness constants are denoted by using \(k_t\) and \(K_r\). For sake of simplification, two special cases are considered. One case is that a simply supported plate with uniform rotational spring stiffness along each edge. The other is that the translational edge spring stiffness varies while the rotational spring stiffness is infinitely large.

The first case we consider is that a simply supported annular sector plate with elastic restraints against edge rotations. The effect results under different rotational spring stiffness are shown in Fig. 5. It can be seen that the first few natural frequencies are sensitive to the stiffness values of the restraining springs, indicating that changing boundary conditions constitutes an effective way of modifying the number of resonant peaks. But the rotational spring stiffness is not sensitive to the amplitude of vibration response of annular sector plates in the lower frequency range. With the rotation restraint further increasing, the peaks of the velocity mobility are shifted toward higher frequency. The speed of the effect is rapid when rotational spring stiffness is relatively small.
For the second case considered in this calculation, the translation spring stiffness values varies from $10^3$ to $10^{12}$, while setting $K_r$ to infinite for each case. The velocity mobility results are plotted in Fig. 6. It can be found that this type of edge spring has the same effect trend compared with that of the rotational type. For the case in which the translation spring is very soft, the structure mobility response has more resonance peaks in low frequency range. It is obvious that the transfer point mobility is more complicated than the driving point mobility. Compared with the first case, it can be noticed that there are more modal frequencies in the specific frequency range in this case. Therefore, the translation spring stiffness is more sensitivity to the modal frequency than the rotation spring stiffness.

![Graphs showing velocity mobility for different values of $K_r$ and $k_t$.](image)

**FIGURE 4.** Velocity mobility of two positions on annular sector plate (SSSS)

**FIGURE 5.** Velocity mobility of two positions on plate for different values of $K_r$

**FIGURE 6.** Velocity mobility of two positions on plate for different values of $k_t$

**CONCLUSIONS**

An analytical method is described for the vibration analysis of annular sector plates with general elastic restraints along each edge, which allows treating all the classical homogenous boundary conditions as the special cases when the stiffness for each of the restraining spring is equal to either zero or infinity. Regardless of boundary conditions,
the displacement function is invariantly expressed as an improved trigonometric series which converges uniformly at an accelerated rate. Since the displacement solution is constructed to have $C^3$ continuity, the current solution, although sought in a weak form from the Rayleigh–Ritz procedure, is mathematically equivalent to a strong solution which simultaneously satisfies both the governing differential equation and the boundary conditions on a point-wise basis.

Free and forced vibrations can be easily solved through simple matrix calculation. Several numerical examples are presented to demonstrate the accuracy and reliability of the current method for various boundary conditions. It has been shown that the stiffness of the translation and rotation restraining springs along each edge can have significant effects on the modal behavior and the vibration response of annular sector plates. With the increase of boundary supporting stiffnesses, the effects on modal characteristic are mainly reflected by lower modes; however, the variation of vibration responses in the higher frequencies may be larger than those in the lower frequencies.

In addition to providing a unified solution to annular sector plates under a wide range of boundary conditions, the current method can be universally applied to any sector angle up to $2\pi$. Although the stiffness for each restraining spring is here assumed to be uniform, any non-uniform, discrete, or partial stiffness distribution can be readily considered by accordingly modifying potential energies given in Eq. (5).

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REFERENCES