When sound propagates through a random media, the wave properties of the acoustic field may be viewed as stochastic variables. It is thus natural to investigate the relationship between the statistical properties of the acoustic field and the random fluctuations of the waveguide. The interest here is a 2-D ocean waveguide with a rough seabed. Numerical solutions to the 2-way integral coupled mode equations (ICME), for random realizations of the roughness from a wavenumber power spectrum, provide the statistics of the modal intensity and cross-mode coherence with range. The roughness induces mode coupling within the trapped spectrum, between the trapped and the continuum spectrum, and to the back propagating modal spectrum. Instead of a master equation for the modal intensities to study the connection between the statistics of the acoustics and the fluctuations in the waveguide as has been advocated in previous studies, the conservation law for acoustic energy flux is used to develop an expression for the individual modal Poynting vectors. In addition to exact numerical computations of the range derivatives of the modal energy flux vectors for both forward and backward propagation, a Poynting vector master equation is derived for the case where the Born approximation is valid.
INTRODUCTION

When sound propagates through a random ocean waveguide, the wave properties of the acoustic field may be viewed as stochastic variables. It is thus natural to investigate the relationship between the statistical properties of the acoustic field and the random fluctuations of the waveguide. Dozier and Tappert derived a master equation for the ensemble averaged modal intensities in a deep-water waveguide that possessed inhomogeneities in the water column as a result of the presence of diffuse linear internal waves. [1] Such stochastic equations have the form of a coupled set of diffusion equations. Also for the diffuse internal wave problem, Colosi and Morozov derived an equation for the ensemble averaged cross-mode coherence matrix. [2] The cross-mode coherence is the more general quantity of interest since the diagonal components of this matrix are just the modal intensities.

There are three motivating factors for the present work. First, previous studies have ignored the backscattered field. Thus, for example, the previously derived stochastic equations cannot be used to address the statistics of reverberation. The expressions derived include the effects of backscattering and scattering into the modal continuum. Second, these previous studies ignored the modal continuum. For example, in the scattering from a random rough seabed surface in shallow water, significant amounts of energy can be scattered to higher vertical angles and into the modal continuum. Third, for a stochastic equation to be useful, a numerical methodology is required. The approach taken in the current work relies on a Born series approach to obtain expressions for the ensemble average of the cross mode coherence in terms of the incident acoustic field and averages of products of the coupling matrices.

FORMULATION

Integral coupled mode equations

The acoustic pressure expressed in terms of a normal mode expansion is

\[ P(x, z) = \sum_n R_n(x)\phi_n(x, z) \]  

where \( R_n(x) \) and \( \phi_n(x, z) \) are the modal amplitudes and depth eigenfunctions, respectively. The depth functions are local in \( x \) and satisfy

\[ \left( \frac{d^2}{dz^2} + k(x, z)^2 - k_n(x, z)^2 \right) \phi_n(z, x) = 0 \]  

and the modal amplitudes satisfy [3]

\[ |R^+\rangle = \hat{G}^+ |\rho\rangle + \hat{G}^+ \hat{C} |R^-\rangle \]  

\[ |R^-\rangle = \hat{G}^- |\rho\rangle + \hat{G}^- \hat{C} |R^+\rangle \]  

where \( |\rho\rangle \) (not to be confused with the density) is the source vector; \( \rho_n = \frac{1}{d} \phi_n(z_0) \), and \( d(z_0) \) and \( z_0 \) are the density and the source depth, respectively.

For the applications considered in this study the source is placed at \( x = 0 \) and the coupling is zero for \( x < 0 \). Thus for \( x > 0 \), \( \hat{G}^- |\rho\rangle = 0 \). The modal amplitudes and source functions are, respectively

\[ |R^\pm\rangle = [R_1^\pm R_2^\pm \cdots R_{N-1}^\pm R_N^\pm]^\dagger \]  

\[ |\rho\rangle = [\rho_1 \rho_2 \cdots \rho_{N-1} \rho_N]^\dagger \]
and the mode coupling matrix operator, $\hat{C}$, and the Green’s function matrix operator in the absence of mode coupling, $\hat{G}$, are, respectively

$$
\hat{C} = \begin{pmatrix}
C_{11} & C_{12} & \cdots & C_{1N-1} & C_{1N} \\
C_{11} & C_{12} & \cdots & C_{1N-1} & C_{1N} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
C_{N-11} & C_{N-12} & \cdots & C_{N-1N-1} & C_{N-1N} \\
C_{N1} & C_{N2} & \cdots & C_{NN-1} & C_{NN}
\end{pmatrix}
$$

(7)

$$
\hat{G}^\pm = \begin{pmatrix}
\hat{G}_1^\pm & 0 & \cdots & 0 & 0 \\
0 & \hat{G}_2^\pm & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \hat{G}_{N-1}^\pm & 0 \\
0 & 0 & \cdots & 0 & \hat{G}_N^\pm
\end{pmatrix}
$$

(8)

**Poynting vector**

Of significant importance in wave propagation problems is the concept of energy flux. The time averaged Poynting vector is

$$
S = \frac{i}{4\omega \rho(z)} [P \nabla P^* - P^* \nabla P].
$$

(9)

Insertion of the normal mode expansion gives

$$
S = \frac{i}{4\omega d^2(2\omega) d(z) \hat{f}} \sum_m \sum_n \zeta_{mn}
$$

(10)

where

$$
\zeta_{mn} = [R_m \dot{R}_n^* \phi_m \phi_n^* - cc] + [R_m R_n^* \phi_m \dot{\phi}_n^* - cc]
$$

(11)

and $cc$ signifies complex conjugate of the preceding term with $\hat{f} = \frac{d}{dx} f$. In terms of forward and back-going modal amplitudes the cross-mode coherence and the cross-mode derivative coherence are

$$
R_m R_n^* = (R_m^+)^* (R_n^+) + (R_m^-)^* (R_n^-) + (R_m^-)^* (R_n^+) + (R_m^+)^* (R_n^-)
$$

(12)

and

$$
R_m R_n^* = (R_m^+)^* (R_n^+) + (R_m^-)^* (R_n^-) + (R_m^+)^* (R_n^-) + (R_m^-)^* (R_n^+)
$$

(13)

Desired is an approximate expression that relates $\frac{d}{dx} <S>$ (where $<>$ denotes ensemble average) directly to ensemble averages of products of the coupling matrices. To achieve such an expression we first write $\frac{d}{dx} S$ as

$$
\frac{d}{dx} S = \frac{1}{4\omega d^2(2\omega)} \hat{f} \sum_m \sum_n \left\{ \frac{d}{dx} [R_m \dot{R}_n^* \phi_m \phi_n^*] + [R_m R_n^*] \frac{d}{dx} \phi_m \phi_n^* - cc + \frac{d}{dx} [R_m \dot{R}_n^* \phi_m \phi_n^*] + [R_m R_n^*] \frac{d}{dx} \phi_m \phi_n^* - cc \right\}
$$

(14)
Further, assuming \( < [R_m R_n^*] [\phi_m \phi_n^*] > = < [R_m R_n^*] > < [\phi_m \phi_n^*] > \) where \( <> \) denotes ensemble average, one has

\[
\frac{d}{dx} < S > = \frac{1}{4 \omega \rho_0^2 \rho(z)} k \sum_m \sum_n \frac{d}{dx} < [R_m R_n^*] > < [\phi_m \phi_n^*] > + < [R_m R_n^*] > \frac{d}{dx} < [\phi_m \phi_n^*] > - cc + \frac{d}{dx} < [R_m R_n^*] > < [\phi_m \phi_n^*] > + < [R_m R_n^*] > \frac{d}{dx} < [\phi_m \phi_n^*] > - cc . \tag{15}
\]

The important terms to compute are \( < [R_m R_n^*] > \) and \( < [R_m R_n^*] > \) because the physics that drives the fluctuations of the acoustic field are the fluctuations of the mode coupling matrices, and it these terms that contain ensemble averages of products of various orders of the coupling matrices. The derivatives of these terms can easily be obtained numerically. The following work discusses how these various ensemble averages can be computed using a Born approximation approach.

**Born expansion of modal amplitudes**

The main goal of this study is to directly relate the statistics of the acoustic field to the statistics of the fluctuations of the physical properties of the waveguide. If the random fluctuations are small enough, then the modal amplitudes can be expanded in a Born series. It is assumed that there exists a background waveguide whose features do not induce mode coupling, and thus the variability of the horizontal wavenumber eigenvalues and depth mode functions as a result of the random fluctuations of the waveguide can be neglected. This is analogous to the distorted wave Born approximation (DWBA) used to describe the unperturbed or incident field prior to scattering in direct nuclear reaction theory. This allows such objects as the background Green's function, the horizontal wavenumber eigenvalues, and the normal mode depth functions to be considered independent of a particular waveguide realization and thus these quantities can be removed from the ensemble averaging. Thus, it is assumed that the acoustical quantities that appear in Eq. 2 are unaffected by the statistical fluctuations of the waveguide. Further, this means that the Greens functions \( G^+ \) and \( G^- \) are unaffected by the fluctuations. For example, in the case where there is a background range dependence of the waveguide, the Green's function that may be viewed as adiabatic is

\[
\hat{G}_n^+(x,x')f(x') = - \int_0^x dx' \frac{i}{2k_n(x')} \exp(i \gamma_n(x)) \exp(-i \gamma_n(x')) f(x') , \tag{16}
\]

and

\[
\hat{G}_n^-(x,x')f(x') = - \int_x^\infty dx' \frac{i}{2k_n(x')} \exp(-i \gamma_n(x)) \exp(i \gamma_n(x')) f(x') , \tag{17}
\]

where

\[
\gamma_n(x) = \int_0^x k_n(x') dx' . \tag{18}
\]

If the background waveguide is flat and thus range-independent then

\[
\hat{G}_n^+(x,x')f(x') = - \int_0^x dx' \frac{i}{2k_n} \exp(\pm ik_n x) \exp(\mp ik_n x') . \tag{19}
\]

Thus, it is the statistics of modal amplitudes that characterize the statistics of the acoustic field. Specifically it is a perturbative treatment of Eqs. 3-4 that allows for a derivation of relations that relate the statistics of the waveguide to those of the acoustic field. In anticipation...
of keeping coupling terms up to second order in the cross-mode coherence, Eqs. 3-4 may be rewritten as

\[
| R^+ \rangle = \hat{G}^+ | \rho \rangle + \hat{G}^+ \hat{C} | \hat{G}^+ \rho \rangle + \hat{G}^+ \hat{C} \hat{G}^+ \hat{C} | R^+ \rangle + \\
\hat{G}^+ \hat{C} \hat{G}^- \hat{C} | R^+ \rangle + \hat{G}^+ \hat{C} \hat{G}^- \hat{C} | R^- \rangle + \hat{G}^+ \hat{C} \hat{G}^+ \hat{C} | R^- \rangle
\]

(20)

\[
| R^- \rangle = \hat{G}^- \hat{C} \hat{G}^+ | \rho \rangle + \hat{G}^- \hat{C} \hat{G}^+ \hat{C} | R^+ \rangle + \\
\hat{G}^- \hat{C} \hat{G}^- \hat{C} | R^+ \rangle + \hat{G}^- \hat{C} \hat{G}^- \hat{C} | R^- \rangle.
\]

(21)

To make theoretical progress the highly oscillatory range dependence is removed from \( R_n^\pm \) by introducing \( \Psi_n^\pm(x) \) as \( \Psi_n^\pm(x) = \exp(\mp i \kappa_n x) R_n^\pm(x) \) and

\[
\Theta_n^\pm = \exp(\mp i \kappa_n x) \hat{G}_n^\pm \rho_n.
\]

Note that \( \Theta_n^- = 0 \) for \( x > 0 \).

The coupled equations for \( \Psi_n^\pm \) are

\[
\Psi_n^+ = \Theta_n^+ - \frac{i}{2 \kappa_n} \int_{0}^{x} dx' \sum_{p} \Theta_p^+(x') C_{np}(x') \exp(i (L_{pn}) x') + \\
\frac{-i}{2 \kappa_n} \int_{0}^{x} dx' \int_{0}^{x} dx'' \sum_{p} \frac{-i}{2 \kappa_p} C_{np}(x') C_{pq}(x'') \exp[ i (L_{pn}) x' + i (L_{pq}) x''] \Psi_q^+(x'') + \\
\frac{-i}{2 \kappa_n} \int_{0}^{x} dx' \int_{x'}^{\infty} dx'' \sum_{p} \frac{-i}{2 \kappa_p} C_{np}(x') C_{pq}(x'') \exp[ i (L_{pn}) x' + i (K_{pq}) x''] \Psi_q^+(x'') + \\
\frac{-i}{2 \kappa_n} \int_{0}^{x} dx' \int_{0}^{x} dx'' \sum_{p} \frac{-i}{2 \kappa_p} C_{np}(x') C_{pq}(x'') \exp[ i (L_{pn}) x' - i (K_{pq}) x''] \Psi_q^-(x'') + \\
\frac{-i}{2 \kappa_n} \int_{0}^{x} dx' \int_{0}^{x} dx'' \sum_{p} \frac{-i}{2 \kappa_p} C_{np}(x') C_{pq}(x'') \exp[ i (L_{pn}) x' - i (K_{pq}) x''] \Psi_q^-(x'')
\]

(23)

where \( L_{ab} = k_a - k_b \) and \( K_{ab} = k_a + k_b \). Also,

\[
\Psi_n^- = - \frac{i}{2 \kappa_n} \int_{0}^{\infty} dx' \sum_{p} \Theta_p^+(x') C_{np}(x') \exp(i (K_{pn}) x') + \\
\frac{-i}{2 \kappa_n} \int_{0}^{x} dx' \int_{0}^{x} dx'' \sum_{p} \frac{-i}{2 \kappa_p} C_{np}(x') C_{pq}(x'') \exp[ i (K_{pn}) x' + i (L_{pq}) x''] \Psi_q^+(x'') + \\
\frac{-i}{2 \kappa_n} \int_{0}^{x} dx' \int_{0}^{x} dx'' \sum_{p} \frac{-i}{2 \kappa_p} C_{np}(x') C_{pq}(x'') \exp[ i (K_{pn}) x' - i (K_{pq}) x''] \Psi_q^+(x'') + \\
\frac{-i}{2 \kappa_n} \int_{0}^{x} dx' \int_{0}^{x} dx'' \sum_{p} \frac{-i}{2 \kappa_p} C_{np}(x') C_{pq}(x'') \exp[ i (L_{pn}) x' + i (K_{pq}) x''] \Psi_q^-(x'') + \\
\frac{-i}{2 \kappa_n} \int_{0}^{x} dx' \int_{0}^{x} dx'' \sum_{p} \frac{-i}{2 \kappa_p} C_{np}(x') C_{pq}(x'') \exp[ i (L_{pn}) x' - i (K_{pq}) x''] \Psi_q^-(x'')
\]

(24)

**Special case for neglect of backward components**

Before proceeding to the full expression for the cross mode coherence that includes both forward and backward going components, the ensemble average of the cross mode coherence is first considered for the special case where terms involving \( \Psi_n^- \) and \( G \) are set to zero. Following Dozier [1] and consistent with the idea that the scattering is weak, \( \Psi_n^+(x'') \) is approximated with \( \Psi_n^+(x) \) allowing one to remove \( \Psi_n^- \) from the integrand in the integrals over \( x'' \) that arise in
forming $\langle \Psi_n^+ (\Psi_m^+)^* \rangle$ from Eqs. 23-24. Further, statistical independence of the coupling and the acoustic field is assumed; specifically, that

$$< S_{ab} S_{cd} \Theta_e^* \Theta_f^* >=< S_{ab} S_{cd} >< \Theta_e^* \Theta_f^* >$$

and

$$< S_{ab} S_{cd} \Theta_e^* \Psi_f^+ >=< S_{ab} S_{cd} >< \Theta_e^* \Psi_f^+ >$$

(25)

(26)

where $< A >$ is the ensemble average of $A$. One should note that the first non-zero term in the ensemble average that contains the coupling matrices is second order since $< C >= 0$. Another assumption is that

$$< C_{np}(x') C_{pq}(x'') > =< C_{np}(x') C_{pq}(\xi) >$$

(27)

with $\xi = x' - x''$. In this assumption the ensemble average of the correlation coupled mode matrix is slowly varying with respect to $x'$ and can have a strong dependence on $\xi$. The contribution to the integrals over the coupled mode correlation become small when $\xi$ exceeds a correlation distance. These two assumptions allow the integrals $\int_0^x dx''$ and $\int_0^{x'} dx''$ to be replaced with $\int_0^\infty d\xi$ with the appropriate changes in the integrand.

Using the weak scattering limit and the assumed form of the mode coupling correlation matrix one can show that

$$< \Psi_n^+ (\Psi_m^+)^* >=< \Theta_n^+ (\Theta_m^+)^* > + \frac{i}{2k_p^* 2k_m^*} < \Theta_n^+ (\Psi_q^+)^* > C_{mp}(x') C_{pq}(x'') \exp[-i (L_p^* x' - i(\xi) x'') +$$

$$+ \sum_p \sum_q \int_0^x dx' \int_0^x dx'' - \frac{i}{2k_n^* 2k_p^*} < \Theta_n^+ (\Theta_q^+)^* > C_{np}(x') C_{mq}(x'') \exp[i (L_m^* x' - i(\xi) x'')]$$

(28)

where the notation $cc m \rightarrow n$ signifies a term that is the complex conjugate of the previous term with the indices $m$ and $n$ interchanged. Further, defining

$$X_{npmq}(x') = \int_0^\infty d\xi < C_{np}(x') C_{mq}(\xi) > \exp(i L_{qm}^* \xi)$$

(29)

the cross-mode coherence for the case that the backward components are neglected become

$$< \Psi_n^+ (\Psi_m^+)^* >=< \Theta_n^+ (\Theta_m^+)^* > + \frac{i}{2k_p^* 2k_m^*} < \Theta_n^+ (\Psi_q^+)^* > X_{mppq}(x') \exp[-i (L_p^* x' + L_q^* x') +$$

$$+ \sum_p \sum_q \int_0^x dx'$$

(30)

Equation 30 can be approximately solved by the Born approximation by replacing $\Psi_q^+$ on the RHS with $\Theta_q^+$ which gives

$$< \Psi_n^+ (\Psi_m^+)^* >=< \Theta_n^+ (\Theta_m^+)^* > + \frac{i}{2k_p^* 2k_m^*} < \Theta_n^+ (\Theta_q^+)^* > X_{mppq}(x') \exp[-i (L_p^* x' + L_q^* x') +$$

$$+ \sum_p \sum_q \int_0^x dx'$$

(31)
If on the other hand one follows Dozier [1], and \( \Theta_q^+ \) is replaced with \( \Psi_q^+ \) on the RHS of Eq. 30 then

\[
< \Psi_n^+(\Psi_m^+)^* > = < \Theta_n^+(\Theta_m^+)^* > + \sum_p \sum_q \int_0^\infty dx' \left\{ \frac{i}{2k_p} \frac{i}{2k_m} < \Psi_n^+(\Psi_q^+)^* > X_{mppq}(x') \exp[-i(L_{pm}^* + L_{qp}^*)x'] \right. \\
+ \text{cc } m \rightarrow n \\
\left. + \frac{i}{2k_n} \frac{i}{2k_m} < \Psi_p^+(\Psi_q^+)^* > X_{nmpq}(x') \exp[-i(L_{np} + L_{qm}^*)x'] \right\}. \tag{32}
\]

If \( L_{ab} = L_{ab}^* \) and \( m = n \) and \( < \Psi_p^+(\Psi_q^+)^* > \) is approximated by the random phase approximation [1], because of the rapid oscillations in the integrand, as \( < \Psi_p^+(\Psi_q^+)^* > = < \Psi_p^+(\Psi_q^+)^* > \delta_{pq} \) then we obtain

\[
< |\Psi_n^+|^2 > = < |\Theta_n^+|^2 > + \frac{1}{4k_n^2} \sum_p < |\Psi_p^+|^2 > D_{nppn}(x) - \frac{2k_n}{k_p} < |\Psi_n^+|^2 > D_{nppn}(x) \tag{33}
\]

where

\[
D_{abcd}(x) = \int_0^\infty dx' X_{abcd}(x'). \tag{34}
\]

Eq. 33 is a stochastic integral equation for the forward propagating mode intensity where the back propagating component has been neglected. Like Eq. 30, Eq. 33 can be solved with the Born approximation by replacing \( < |\Psi_n^+|^2 > \) on the RHS with \( < |\Theta_n^+|^2 > \).

**Cross-mode coherence**

The ensemble average of the cross-mode coherence that contains both the forward and backward going components is now constructed. Defining

\[
Y_{nmpq}(x') = - \int_0^\infty d\xi < C_{np}(x')C_{mq}(\xi) > \exp(-iK_{qm}^*\xi) \tag{35}
\]

and

\[
\hat{X}_{nmpq}(x') = \int_0^\infty d\xi < C_{np}(x')C_{mq}(\xi) > \exp(-iL_{qm}^*\xi) \tag{36}
\]

the cross-mode coherence, using the previously stated assumptions is

\[
< \Psi_n^+(\Psi_m^+)^* > = < \Theta_n^+(\Theta_m^+)^* > + \sum_p \sum_q \int_0^\infty dx' \left\{ \frac{i}{2k_m^*} \frac{i}{2k_p^*} < \Theta_n^+(\Theta_q^+)^* > X_{mppq}(x') \exp[-i(K_{pm}^* - K_{qp}^*)x'] \right. \\
+ \text{cc } m \rightarrow n \\
\left. + \frac{i}{2k_n^*} \frac{i}{2k_m^*} < \Theta_n^+(\Theta_q^+)^* > \hat{X}_{mppq}(x') \exp[i(K_{pm}^* - L_{pq}^*)x'] \right\}.
\]
D. Knobles and J. Sagers

\[ <\Psi_n^+(\Psi_m^-)^*> = \sum_{p,q} \left( \int_x^\infty dx' \frac{i}{2k_m} \frac{i}{2k_p} <\Theta_{pq}^+(\Psi_q^-)^* > X_{mpqq}(x') \exp[-i(K_{pm} + L_{qp})x'] + \frac{i}{2k_m} \frac{i}{2k_p} <\Theta_{pq}^+(\Psi_q^-)^* > Y_{pmqp}(x') \exp[-i(K_{pm} - K_{qp})x']x'' \right) \] 

and

\[ <\Psi_n^-(\Psi_m^-)^*> = \sum_{p,q} \left( \int_x^\infty dx' \int_x^\infty dx'' \frac{i}{2k_m} \frac{i}{2k_p} <\Theta_{pq}^+(\Psi_q^-)^* > X_{npmq}(x') \exp[i(K_{pn} - K_{qm})x'] + \right) \] 

The Born approximation is now applied to form a solution for the cross mode coherence. First, \( \Psi_n^+ \) is replaced with \( \Theta_n^+ \) on the RHS of Eqs. 37-39. Second, from Eq. 24 one replaces \( \Psi_n^- \) with \( -\frac{i}{2k_m} \int_x^\infty dx' \sum_p \Theta_{pq}(x') C_{np}(x') \exp(i(K_{pn})x') \). However, one recognizes that these terms become zero because \( <C> = 0 \). Thus,

\[ <\Psi_n^+(\Psi_m^-)^*> = <\Theta_n^+(\Theta_m^-)^*> + \left( \sum_{p,q} \int_x^\infty dx' \frac{i}{2k_m} \frac{i}{2k_p} <\Theta_{pq}^+(\Psi_q^-)^* > X_{mpqq}(x') \exp[-i(L_{pm} + L_{qp})x'] + \right) \]

\[ \sum_{p,q} \int_x^\infty dx' \frac{i}{2k_m} \frac{i}{2k_p} <\Theta_{pq}^+(\Psi_q^-)^* > Y_{mpqp}(x') \exp[i(L_{pm} - L_{qp})x'] + \right) \]

and

\[ <\Psi_n^-\Psi_m^- > = \sum_{p,q} \left( \int_x^\infty dx' \frac{i}{2k_m} \frac{i}{2k_p} <\Theta_{pq}^+(\Psi_q^-)^* > X_{mpqq}(x') \exp[-i(K_{pm} + L_{qp})x'] + \right) \]

\[ \sum_{p,q} \int_x^\infty dx' \frac{i}{2k_m} \frac{i}{2k_p} <\Theta_{pq}^+(\Psi_q^-)^* > Y_{mpqp}(x') \exp[i(K_{pm} - K_{qp})x'] + \right) \]

\[ \sum_{p,q} \int_x^\infty dx' \frac{i}{2k_m} \frac{i}{2k_p} <\Theta_{pq}^+(\Psi_q^-)^* > X_{mpqp}(x') \exp[i(K_{pn} - L_{qm})x'] \] 

and

\[ <\Psi_n^-\Psi_m^- > = \sum_{p,q} \left( \int_x^\infty dx' \frac{i}{2k_m} \frac{i}{2k_p} <\Theta_{pq}^+(\Psi_q^-)^* > X_{npmq}(x') \exp[i(K_{pn} - K_{qm})x'] \right) \]

**DISCUSSION**

This paper addressed the problem of sound propagation through an ocean waveguide that has small random fluctuations. The inhomogeneities induce mode coupling within the trapped
spectrum, between the trapped and the continuum spectrum, and to the back propagating modal spectrum. Expressions were derived for the ensemble average of acoustic quantities such as the modal intensity and the cross mode coherence in terms of ensemble averages of the incident field and products of the mode coupling matrices. It was assumed that the acoustic variability occurs solely in the normal mode amplitudes and thus the horizontal wavenumber eigenvalues and depth mode functions do not vary as a result of the random fluctuations of the waveguide. This allows such objects as the background Green’s function, the horizontal wavenumber eigenvalues, and the normal mode depth functions to be considered independent of a particular waveguide realization and thus these quantities can be removed from the ensemble averaging. It was further assumed that the fluctuations that induce the mode coupling and thus scattering are weak. In such cases a Born approximation can be used to solve the coupled equations for the ensemble averaged cross mode coherence.

Equation 31 is the Born approximation solution for cross mode coherence for the case where backscatter can be neglected. The approach used in its derivation differs from previous studies in that instead of deriving a set of coupled diffusion equations for the cross-mode coherence, a solution is directly constructed from the Born approximation that is consistent with the assumption that the scattering is weak. Numerically one needs to preform the integrations over the ensemble averaged mode coupling matrix correlations in Eqs. 30 and 35. The numerical form of the coupling matrices have been presented elsewhere. An additional complicating feature about the coupling matrices not discussed here is that one of the coupling matrices has a range derivative acting on terms to its right. However, with the weak scattering assumption this results only in taking derivatives of certain exponential terms. Equations 40-42 are the main result of the theoretical approach to stochastic equations that include both forward and backward propagating components. They, like Eq. 31, are solutions to coupled integral equations using the Born approximation.

Equations 40-42 and variants form the basis of the Poynting vector flux law in Eq. 15. Future work will test various aspects of the flux law based on the perturbative results for the ensemble average cross mode coherence.

**References**

