1aUW9. Rayleigh scattering of sound by spherically symmetric bodies

Oleg A. Godin

*Corresponding author's address: CIRES, University of Colorado, Boulder, Colorado 80305, oleg.godin@noaa.gov

An obstacle's shape is often approximated by a sphere in analyses of sound scattering by air bubbles, objects on or near the seafloor, marine organisms, clouds of suspended particles, etc. Here, an asymptotic technique is developed to study low-frequency sound scattering from spherically symmetric inhomogeneous obstacles. The obstacle can be fluid, solid, or a fluid-filled solid shell. Physical properties of the obstacle are arbitrary piece-wise continuous functions of the distance to its center. The radius of the obstacle is assumed to be small compared to the wavelengths of sound in the surrounding fluid as well as of compressional and shear waves inside the obstacle. General properties of the sound scattering by spherically symmetric bodies are established. Resonant Rayleigh scattering is studied in detail. For plane and spherical incident waves, it is discussed which physical and geometrical parameters of the obstacle can be retrieved from the scattered acoustic field.

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DIFFRACTION OF SOUND ON A SPHERICALLY SYMMETRIC OBSTACLE

Consider a spherically-symmetric object of radius $a$ located in a homogeneous, inviscid fluid with sound speed $c_0$ and density $\rho_0$. In a spherical coordinate system $(r, \theta, \varphi)$ with the center $r = 0$ at the center of the obstacle, material parameters of the latter are independent of the polar and azimuthal angles $\theta$ and $\varphi$ and may be arbitrary functions of the radial distance $r$. Using separation of variables, an arbitrary incident continuous acoustic wave can be represented as a sum of “partial waves” with the symmetry consistent of that of the object [Bowman et al., 1969]:

$$ p_n = \sum_{m=-\infty}^{\infty} D_{mn} e^{i\omega m} j_n(kr) j_n(kr). \quad (1) $$

Here and below $p$ stands for acoustic pressure, acoustic wavenumber in the surrounding fluid $k = \omega/c_0$, $j_n(\cdot)$ are associated Legendre functions and $j_n(\cdot)$ are spherical Bessel functions [Abramowitz and Stegun, 1972]. The coefficients $D_{mn}$ define amplitudes of individual “partial waves” (spherical harmonics) and specify the incident wave. The diffracted (scattered) wave is given by the infinite series [Bowman et al., 1969]

$$ p_n = -\sum_{m=-\infty}^{\infty} A_n D_{mn} e^{i\omega m} h_n^{(1)}(kr), \quad r \geq a, \quad \quad (2) $$

where $h_n^{(1)}(z) = j_n(z) + i y_n(z)$, $h_n^{(1)}(z)$ and $y_n(z)$ are spherical Hankel and Neumann functions, and coefficients $A_n$ describe scattering of individual spherical harmonics.

Using equations (1) and (2) to calculate the power flux through the surface $r = a$ of the obstacle and taking advantage of orthogonality of the surface harmonics $e^{i\omega m} P_n^m(\cos \theta)$ on the sphere, we find that

$$ \text{Re} A_n \geq |A_n|, \quad n = 0, 1, 2, \ldots. \quad (3) $$

In the case of elastic scattering, where there is no energy loss inside the obstacle, equation (3) becomes

$$ \text{Re} A_n = |A_n|. \quad (4) $$

Then equation (2) is known to simplify considerably for a plane incident wave; the leading order of the asymptotics of the scattered field is given by the terms with $n = 0$ and $n = 1$ in the series (2) [Bowman et al., 1969]. The same result is valid for an arbitrary incident wave as long as either $a \sqrt{p_{in}} \ll 1$ at $r \leq a$ or the observation point is in the far field [Godin, 2013]. Under these conditions, knowledge of two scattering amplitudes, $A_0$ and $A_1$, is sufficient to fully characterize Rayleigh scattering by a particular obstacle. On the contrary, higher-order spherical harmonics contribute materially to $p_{sc}$ in the near field, when a point source is located at a distance $b = a O(1)$ from the center of the obstacle [Godin, 2011, 2013]. Under these circumstances, knowledge of many scattering amplitudes is required to characterize the obstacle and calculate the wave field.

SCATTERING FROM A FLUID OBSTACLE

In the particular case, when the object is an inhomogeneous fluid sphere, acoustic pressure inside the object can be represented by a series similar to equations (1) and (2):

$$ p_i = -\sum_{n=0}^{\infty} \sum_{m=-n}^{n} D_{mn} e^{i\omega m} P_n^m(\cos \theta) f_n(r), \quad 0 \leq r \leq a. \quad (6) $$

The functions $f_n(r)$ describe the radial dependence of the acoustic pressure in individual spherical harmonics, are finite everywhere inside the obstacle, and satisfy one-dimensional reduced wave equations

$$ \frac{d}{dr} \left( \frac{r^2}{\rho(r)} \frac{df_n(r)}{dr} \right) + \frac{k^2(r)r^2-n(n+1)}{\rho(r)} f_n(r) = 0, \quad (7) $$

where $k(r) = \omega/c(r)$. Within the scatterer, the sound speed $c(r)$ and density $\rho(r)$ are assumed to be piece-wise continuous functions of $r$. Acoustic fields inside and outside the obstacle are related by the boundary conditions

$$p_i = p_{in} + p_{o}, \quad \frac{\partial p_i}{\partial r} = M\left(\frac{\partial p_{in}}{\partial r} + \frac{\partial p_{o}}{\partial r}\right), \quad r = a; \quad M = \frac{\rho(a)}{\rho_0}. \quad (8)$$

Denote

$$S_n = \frac{Mf_n}{\alpha df_n/dr}_{r=a}. \quad (9)$$

The quantity $S_n$ has the meaning of a dimensionless impedance associated with $n$-th spherical harmonics. It follows from equation (7) that the impedances have the following frequency dependence:

$$S_n = S_n^{(0)} + k_o^2a^2S_n^{(2)} + \ldots, \quad n \geq 1, \quad (10)$$

provided $ka \ll 1$.

Using equations (1), (2), and (6) – (9), for the scattering amplitude we find

$$A_n = j_n(k_oa) - S_n k_o \alpha n j_n(k_oa) + \frac{1}{2n+1} \frac{1-nS_n^{(0)} + O(k_o^2a^2)}{1 + (n+1)S_n^{(0)} + O(k_o^2a^2)}, \quad n \geq 1. \quad (11)$$

From equations (10) and (11) and asymptotic expansions of spherical Bessel functions for small arguments [Abramovitz and Stegun, 1972] it follows that, unless $(n+1)S_n^{(0)} = -1$,

$$A_n = \left[\frac{2^n n!}{(2n)!}\right]^2 i(k_oa)^{2n+1} \frac{1-nS_n^{(0)} + O(k_o^2a^2)}{(2n+1) + (n+1)S_n^{(0)} + O(k_o^2a^2)}, \quad n \geq 1. \quad (12)$$

Thus, the scattering amplitudes rapidly decrease with $n$ to calculate $A_n$ in the Rayleigh scattering regime, equation (7) has to be solved in the static limit, i.e., at $\omega = 0$. When $n > 0$, the condition that $f_n$ is bounded is equivalent to $f_n(0) = 0$. The solution is readily available, when the density is either constant or piece-wise constant inside the obstacle. In the general case of continuous or piece-wise continuous density variations, the equation has to be solved numerically.

An analytic solution can be obtained for $A_0$. Following [Brekhovskikh and Godin, 1998], we reduce equation (7) to a Volterra integral equation of the second kind. A successive approximation scheme gives a solution of the integral equations as a power series in $\omega^2$. In particular, we obtain

$$\frac{f_0(r)}{f_0(0)} = 1 - \int_0^r k^2(r) r^2 \frac{\rho(r) dr}{\rho_0 r^2} dr + O(k_o^4a^4) \quad (13)$$

and

$$A_0 = i k_o a e^{-ik_oa} \frac{1 - \langle K \rangle K_0^{-1} \left[1 + O(k_o^{-2})\right]}{3 \left(1 - ik_o a - k_o^2 a^2 \langle K \rangle K_0^{-1} \left[1 + O(k_o^{-2})\right]\right)}. \quad (14)$$

Here $K_0 = \left(\frac{\rho_o c_o^2}{\rho c^2}\right)^{-1}$ is the compressibility of the surrounding fluid, and

$$\langle K \rangle = \frac{3}{4\pi} \int_0^\infty \frac{r^2 dr}{\rho c^2} \quad (15)$$

is the average compressibility of the obstacle. The amplitude $A_0$ (14) of scattering by an inhomogeneous fluid sphere proves to be same as for “effective” homogeneous fluid sphere with compressibility $\langle K \rangle$ and reduces to the hard and soft sphere results, when $\langle K \rangle \rightarrow 0$ and $\langle K \rangle \rightarrow \infty$, respectively [Godin, 2013].

When the obstacle is much more compressible than the surrounding fluid, $\langle K \rangle \gg K_0$, resonance scattering occurs, when $k_o a = \left(3K_0/\langle K \rangle\right)^{1/2} \ll 1$, i.e., at the frequency

$$\omega = \alpha^{-1} \left(\rho_0 \langle K \rangle/3\right)^{1/2}. \quad (16)$$

At the resonance frequency, $|A_0| = 1$ within the accuracy of equation (14). $|A_0|$ reaches its theoretical maximum and increases by a large factor $O(k_o^{-2}a^{-1})$ compared to its off-resonance value.
SCATTERING FROM A SOLID OBSTACLE OR A FLUID-FILLED SHELL

Let a radially-inhomogeneous solid object have density \(\rho(r)\) and Lame parameters \(\lambda(r)\) and \(\mu(r)\). The material parameters of the object are piece-wise continuous functions of the radial distance. In addition to equation (5), we assume that the radius \(a\) of the object is small compared to wavelengths of the shear and compressional waves in the solid. From an asymptotic analysis of the equations of motion of inhomogeneous isotropic solids, we find that the scattering amplitude \(A_1\) will be given by the same equation (14) as for the fluid obstacle if \(\langle K \rangle\) is replaced by the following quantity:

\[
K_s = \frac{3F}{(3\lambda + 2\mu)F + (\lambda + 2\mu)r dF/dr}_{r=a},
\]

where \(F\) is a bounded at \(r \leq a\) solution of the ordinary differential equation

\[
\frac{d}{dr}\left[(\lambda + 2\mu)\left(3F + r \frac{dF}{dr}\right)\right] = 4\frac{d\mu}{dr} F.
\]

The quantity \(u = rF(r)\) has the meaning of the displacement of the particles in the obstacle under action of a uniform static pressure. Equations (17) and (18) also apply when the object has both solid and fluid (\(\mu = 0\)) spherical layers. When \(\mu\) is identically zero, equation (17) reduces to equation (15).

When a thick shell with inhomogeneous \(\lambda(r)\) and \(\mu = \text{const.}\) surrounds an inhomogeneous fluid sphere, by solving equation (18) we find

\[
K_s = \frac{\eta g_1 + (1-\eta)g_2 (1 + 4\mu g_3/3)}{1 - 4\mu/3 (1-\eta)} g_1, \quad g_1 = \frac{3}{a_0^3} \int_0^a r^2 dr, \quad g_2 = \frac{3}{a^3 - a_0^3} \int_0^a r^2 dr, \quad \eta = \frac{a_0^3}{a^3},
\]

Here \(a\) and \(a_0\) are the external and internal radii of the shell; \(g_1\) is the volume average of the compressibility of the inhomogeneous fluid, and \(g_2\) is the volume average of \((\lambda + 2\mu)^{-1}\) in the inhomogeneous shell. In another special case, where the bulk modulus \(B = \lambda + 2\mu/3\) is constant inside the object, equation (17) simplifies and gives \(K_s = 1/B\). When \(K_s\) is large compared to \(K_0\), resonance scattering can occur. With \(\langle K \rangle\) being replaced by \(K_s\), the resonant scattering is described by the same equations as in the fluid obstacle case considered above.

For the scattering amplitude \(A_1\) we find

\[
A_1 = \frac{i}{3} (ka)^3 \frac{1 - \langle M \rangle \left[ 1 + O\left(k_0^2 a^2\right) \right]}{1 + 2 \langle M \rangle \left[ 1 + O\left(k_0^2 a^2\right) \right]},
\]

where \(\langle M \rangle\) is the ratio of the mass of the object to the mass of the surrounding fluid in the volume of the object. This equation remains true for a radially inhomogeneous shell filled with a homogeneous fluid. Equation (18) differs from well-known equations for homogeneous fluid [Anderson, 1950] or solid [Faran, 1951] spheres by only having \(\langle M \rangle\) instead of the ratio \(M\) of the constant densities inside and outside the obstacle. Finally, for the scattering amplitudes \(A_n, n \geq 2\) we find the same expressions as for a homogeneous solid sphere of the same radius. In the Rayleigh scattering regime, these expressions are the same as for a rigid sphere. This fact allows one to extend to spherically symmetric solid obstacles and shells the uniform asymptotic of the spherical wave scattering recently obtained for rigid spheres [Godin, 2013].

CONCLUSION

Any spherically-symmetric inhomogeneity in a fluid scatters low-frequency, plane sound waves like a homogeneous fluid sphere with the compressibility equal to the volume average of the compressibility inside the inhomogeneity. Density of the effective sphere depends on the density distribution in the inhomogeneity and generally is not equal to the volume average of the inhomogeneous density.

Any spherically-symmetric solid body in a homogeneous fluid scatters low-frequency sound waves like an effective homogeneous solid sphere. Mass of the effective solid sphere equals the mass of the body. Bulk modulus of the effective sphere is uniquely defined by the variation of the Lame parameters in the scatterer and has been found explicitly in some particular cases. Shear rigidity of the effective solid sphere can be arbitrary as long as the
wavelength of shear waves is large compared to the radius of the sphere. These results hold also for radially inhomogeneous shells filled with a homogeneous fluid.

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