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1pUW1. A study of the reflection coefficients and backscattering effects of one-dimensional rough poroelastic surfaces using the finite element method

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The reflection and scattering effects of one-dimensional rough poroelastic surfaces are studied using the finite element method. The poroelastic sediment layer is modeled following the classical work of Biot as extended by Stoll, which assumes that two attenuating compressional waves and one attenuating shear wave propagate in the sediment. The rough surfaces are generated using power-law type spectra and the incident wave used is a Gaussian tapered plane wave. This work seeks to assess how the reflection coefficients and backscattering effects of a poroelastic bottom vary as a function of frequency, roughness, and sediment type. Special consideration is given to the mesh required to accurately resolve the effects of the slow compressional and shear waves, which often have wavespeeds slower than the fast compressional wave by an order of magnitude or more. [Work sponsored by the Office of Naval Research, Ocean Acoustics.]

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INTRODUCTION

Accurate modeling of the acoustic interaction with the seafloor is of particular importance in many underwater acoustics problems, such as in the performance assessment of sonar systems. While reflection coefficients can be found quite easily if the interface is assumed flat and if the sediment model used is simplistic, characterizing reflection loss can become significantly more difficult if interface roughness scattering effects are considered and a more robust seafloor sediment model is employed.

Much work has been done toward quantifying the effects of surface roughness on the scattered pressure field. Problems of this type are typically studied using analytic models based on approximations made to the Helmholtz integral formulation. The three most common approximations of this kind are the small-roughness perturbation approximation, the Kirchhoff approximation, and the small-slope approximation [1]. However, while these approximations are useful if their assumptions are met and in their respective ranges of validity, another method, finite element modeling, has been shown to converge to the exact solution of two-dimensional scattering problems and has the added benefit of including the effects of multiple scattering automatically [2].

The acoustic behavior of sediments has also been studied extensively. The earliest models assumed sediments behaved like fluids. However, since sediments generally can support shear stresses, the assumptions governing the use of fluid models are tenuous at best and have been replaced with more robust models, such as those that model the sediment as an elastic or viscoelastic material [1]. However, it has recently been shown that the best fit with experimental reflection data occurs when the sediment is assumed to behave as a poroelastic material [3]. The theory of sound propagation in poroelastic media was first introduced by Biot in a series of classic papers [4]. While a full explanation of Biot's theory is beyond the scope of this paper, it should be stated that poroelastic models allow for three types of waves to propagate in the sediment: two kinds of compressional waves (known as the "fast" and "slow" waves) and one kind of shear wave. This theory was reformulated and extended by Stoll, creating a sediment model that calls for thirteen input parameters [5].

As shown above, roughness scattering effects and models of sediment acoustical behavior have both been given extensive treatment in the literature. However, both issues are seldom considered together. Williams, Gronchocinski, and Jackson studied the combined problem of interface scattering from poroelastic sediments using perturbation theory [6]. The goal of the present paper is to approach problems of this kind using the finite element method.

FINITE ELEMENT METHOD

Overview

The finite element method (FEM) is a numerical technique for approximating solutions to boundary-value problems. FEM requires that the domain of a given problem be discretized into a finite number of subdomains, or elements. A weak formulation of a given partial differential equation is derived, solved in each element, and assembled to generate the global solution [7]. The method is particularly powerful in that a FEM solution converges to the exact solution provided the discretization of elements is sufficiently fine. For this reason, FEM has been used to solve a large variety of problems in many areas of engineering, mathematics, and physics.
Mixed Formulation

Both Biot's original theory and the extension by Stoll have poroelasticity equations formulated in terms of the displacement of the sediment frame and the relative displacement of the pore fluid with respect to the frame. Equations formulated in this way require the solution of two displacement fields and are thus cumbersome to compute using numerical methods such as the finite element method. For this reason, Atalla, Hamdi, and Panneton reformulated the poroelasticity equations in terms of the frame displacement and the pressure field [8]. A formulation like theirs is employed in the commercially available finite element solver used in the present paper [9].

INPUT PARAMETERS

Biot-Stoll Parameters

As briefly mentioned above, Stoll's formulation of Biot theory calls for a sediment model with thirteen input parameters. A table of these parameters is below.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Porosity ($\beta$)</td>
<td>–</td>
</tr>
<tr>
<td>Sediment grain density ($\rho_r$)</td>
<td>kg/m$^3$</td>
</tr>
<tr>
<td>Pore fluid density ($\rho_f$)</td>
<td>kg/m$^3$</td>
</tr>
<tr>
<td>Sediment grain bulk modulus ($K_r$)</td>
<td>GPa</td>
</tr>
<tr>
<td>Pore fluid bulk modulus ($K_f$)</td>
<td>GPa</td>
</tr>
<tr>
<td>Permeability ($\kappa$)</td>
<td>$\mu$m$^2$</td>
</tr>
<tr>
<td>Pore fluid viscosity ($\eta$)</td>
<td>kg/m-s</td>
</tr>
<tr>
<td>Structure factor or tortuosity ($\tau$)</td>
<td>–</td>
</tr>
<tr>
<td>Frame shear modulus ($\mu$)</td>
<td>GPa</td>
</tr>
<tr>
<td>Shear log decrement ($\delta\mu$)</td>
<td>–</td>
</tr>
<tr>
<td>Frame bulk modulus ($K_b$)</td>
<td>GPa</td>
</tr>
<tr>
<td>Frame log decrement ($\delta K$)</td>
<td>–</td>
</tr>
</tbody>
</table>

A full description of what these parameters represent and how they are obtained is beyond the scope of this paper. More information can be found in Ref. 10.

If the frame shear and bulk moduli are allowed to be expressed as complex quantities, the number of input parameters can be decreased to eleven by combining these moduli and their corresponding log decrements using the following relations [10]:

$$\mu = \mu + i \frac{\delta\mu}{\pi}$$

(1)

$$K_b = K_b + i \frac{\delta K}{\pi}K_b$$

(2)

Needed Parameters

The commercial finite element solver used in the present paper requires a slightly different parameter set [9]. Ten of these parameters are either the same or dependent on the parameters required in the Biot-Stoll model. A table of these parameters is below: Seven of the parameters (porosity, pore fluid density, permeability, pore fluid viscosity, tortuosity, complex frame shear modulus, complex frame bulk modulus) are the same as seven of the final eleven Biot-Stoll
TABLE 2: Ten FEM Input Parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Porosity ($\beta$)</td>
<td>–</td>
</tr>
<tr>
<td>Drained density ($\rho_d$)</td>
<td>kg/m$^3$</td>
</tr>
<tr>
<td>Pore fluid density ($\rho_f$)</td>
<td>kg/m$^3$</td>
</tr>
<tr>
<td>Biot-Willis coefficient ($\alpha$)</td>
<td>–</td>
</tr>
<tr>
<td>Pore fluid compressability ($\chi$)</td>
<td>GPa$^{-1}$</td>
</tr>
<tr>
<td>Permeability ($\kappa$)</td>
<td>$\mu$m$^2$</td>
</tr>
<tr>
<td>Pore fluid viscosity ($\eta$)</td>
<td>kg/m-s</td>
</tr>
<tr>
<td>Tortuosity ($\tau$)</td>
<td>–</td>
</tr>
<tr>
<td>Complex frame shear modulus ($\mu$)</td>
<td>GPa</td>
</tr>
<tr>
<td>Complex frame bulk modulus ($K_b$)</td>
<td>GPa</td>
</tr>
</tbody>
</table>

parameters described above. The remaining three parameters can be derived from the Biot-Stoll parameters using the following relation:

$$\rho_d = (1 - \beta)\rho_r$$  \hspace{1cm} (3)

$$\alpha = 1 - \frac{K_b}{K_f}$$  \hspace{1cm} (4)

$$\chi = \frac{1}{K_f}$$  \hspace{1cm} (5)

One final parameter needed is the characteristic frequency that demarcates the change from the low frequency range to the high frequency range as originally discussed by Biot [4]. The characteristic frequency is calculated as follows:

$$f_c = \frac{\eta}{2\pi\rho_f\alpha^2}$$  \hspace{1cm} (6)

where $\alpha$ is the pore size parameter given in meters, which is equal to the pore radius in the case of circular pores.

INTERFACE COUPLING CONDITIONS

In order to accurately model the interaction of an incident pressure field with the seafloor, the finite element subdomain associated with the fluid halfspace must be coupled with the finite element subdomain associated with the poroelastic sediment. This coupling is done by ensuring the continuity of normal stresses, the continuity of relative mass flux, and the continuity of pressure across the interface [11].

Continuity of Normal Stress

The first coupling condition ensures the continuity of normal stress across the interface of the water with the poroelastic sediment and is given below [11]:

$$\sigma_t \hat{n} = -p_a \hat{n}$$  \hspace{1cm} (7)

where $\sigma_t$ is the total stress tensor of the poroelastic material, $p_a$ is the pressure of the water, and $\hat{n}$ is the normal vector pointing away from the poroelastic sediment.
It is helpful to express this equation not in terms of the normal stress of the poroelastic material but in terms of the normal stress of the frame. These two quantities are related by the following expression [12]:

$$\sigma_t \hat{n} = \sigma_s \hat{n} - \beta p_f \hat{n}$$  \hspace{1cm} (8)

where $\sigma_s$ is the stress tensor of the frame and $p_f$ is the pressure of the pore fluid.

**Continuity of Relative Mass Flux**

The second coupling condition ensures the continuity of the relative mass flux across the interface and is given below [11]:

$$u_n + w_n = \frac{1}{\rho_0 \omega^2} \frac{\partial p_a}{\partial n}$$  \hspace{1cm} (9)

where $u_n$ is the normal displacement of the frame, $w_n$ is the normal displacement of the pore fluid relative to the frame, $\rho_0$ is the density of the water, and $\omega$ is the natural frequency.

In order for this condition to be implemented in the mixed formulation being used, the left hand side of Eq. (9) must be expressed purely in terms of the frame displacement and the pore fluid pressure. This reformulation can be achieved by using the following relation [9]:

$$w_n = \frac{1}{\omega^2 \rho_c(\omega)} \left( \frac{\partial p_f}{\partial n} - \rho_f \omega^2 u_n \right)$$  \hspace{1cm} (10)

where $\rho_c(\omega)$ is the complex density given by:

$$\rho_c(\omega) = \frac{\tau}{\kappa} \rho_f + \frac{\eta}{i\omega \beta}$$  \hspace{1cm} (11)

**Continuity of Pressure**

The third coupling condition ensures the continuity of pressure across the interface and is given simply as follows [11]:

$$p_f = p_a$$  \hspace{1cm} (12)

**SURFACE ROUGHNESS**

Random rough surfaces, like those found on the ocean bottom, are generally described statistically in terms of their deviation from a smooth reference surface [13]. The most commonly employed statistical measure of this kind used to describe seafloor roughness is the power spectral density. One simple and commonly used model assumes the seafloor roughness follows power-law spectra, formally described below [1]:

$$W(K) = \frac{w_2}{K^{\gamma_2}}$$  \hspace{1cm} (13)

where $K$ is the spatial wave vector, $K$ is the spatial wavenumber, and $w_2$ and $\gamma_2$ are parameters referred to as the spectral strength and the spectral exponent, respectively.

While power law spectra have been found to agree closely with experimentally measured seafloor roughness in many cases, the fact that the power-law spectral form behaves pathologically when the spatial wavenumber approach zero or infinity limits its utility. To address this limitation, the von Karman spectral form is often used instead. While behaving similarly to the power law in general, the von Karman spectral form approaches a constant
value as the wavenumber approaches zero, and the spectral form approaches zero as the wavenumber approaches infinity \[1\]. Formally, it is given as follows:

\[
W(K) = \frac{w_2}{(K^2 + K_0^2)^{3/2}}
\] (14)

where the parameter \(K_0\) is referred to as the cutoff wavenumber.

Once a spectral form is chosen, surface realizations can then be created for a set of \(N\) points with spacing \(\Delta x\) over length \(L = N\Delta x\). The realizations of the surface height function are then generated as follows for points \(x_n = n\Delta x (n = 1, \ldots, N)\) \[14\] :

\[
f(x_n) = \frac{1}{L} \sum_{j=-N/2}^{N/2-1} F(K_j)e^{iK_jx_n}
\] (15)

where, for \(j \geq 0\),

\[
F(K_j) = [2\pi L W(K_j)]^{1/2} \begin{cases} 
\frac{N(0,1) + iN(0,1)}{\sqrt{2}}, & j \neq 0, N/2 \\
N(0,1), & j = 0, N/2
\end{cases}
\] (16)

and, for \(j < 0\),

\[
F(K_j) = F(K_{-j})^*
\] (17)

In the above expressions, \(K_j = \frac{2\pi j}{L}\) and \(N(0,1)\) represents an independent sample of a zero mean, unit variance Gaussian distribution.

**FURTHER MODELING CONSIDERATIONS**

**Perfectly Matched Layers**

In order to facilitate the modeling of infinite domains using the finite element method, a way to properly truncate the computational domain without negatively affecting the quality of the solution must be implemented. One such technique is the use of perfectly matched layers (or PMLs). Originally formulated by Berenger, PMLs simulate absorbing layers made of an anisotropic damping material that surround the domain of interest and absorb the scattered field present at the boundary. While first formulated for problems in electromagnetism, PMLs have successfully been used in many problems dealing with both acoustic scattering and poroelastic media \[15\].

**Incident Plane Wave**

When modeling infinite domains, it is important to minimize scattering from the ends of surface realizations. This goal can be accomplished by tapering a plane wave so it is of negligible level when it reaches the edge boundaries. One such tapered plane wave is created by applying a Gaussian taper function, producing the following formulation of the incident pressure field \[14\] :

\[
p_{inc}(r) = \exp \left[ i k_{inc} \cdot r - (x - z \cot \theta)^2 / g^2 \right]
\] (18)

where \(k_{inc}\) is the incident wave vector, \(r\) is the position vector, \(\theta\) is the mean grazing angle, and \(g\) is a parameter that controls the tapering.

It should be noted that the Gaussian tapered plane wave given above does not exactly solve the Helmholtz equation; to ensure that the incident pressure field does, the following modified form can be utilized:

\[
p_{inc}(r) = \exp[ik_{inc} \cdot r[1 + w(\textbf{r}) - (x - z \cot \theta)^2 / g^2]}
\] (19)
where

\[ w(r) = \frac{2(x-z \cot \theta)^2 / g^2 - 1}{(kg \sin \theta)^2} \]  

(20)

and \( k \) is the wavenumber of the material through which the incident field propagates (water, in this case).

Much has been written about choosing the right value of the tapering parameter \( g \) [16]. A value of \( g = L/4 \) has been found to work in most cases [14].

**Meshing**

When implementing the finite element method, the way the computational domain is discretized into finite elements, or meshed, is of critical importance. In problems with wave propagation, it is usually wise to create a mesh that is generally uniform and has the maximum element size be no larger than \( \lambda/4 \), where \( \lambda \) is the wavelength. This size restriction allows for proper sampling of the waveform and guards against aliasing. For problems modeling infinite domains, it is also a good idea to have the computational domain be modeled with length \( L = n \lambda \), where \( n \) is some positive integer value. Generally, the larger the value of \( n \) the better, as larger computational domains help negate the impact of possible edge effects. However, when dealing with poroelastic media, one must consider the fact that, as there are three different types of waves, there are three different, often disparate, length scales to consider. To deal with this disparity, the decided length of the computational domain should be in terms of the largest wavelength and the chosen element size should be in terms of the smallest wavelength. While the fast wave always has the largest wavelength by at least an order of magnitude for any given frequency, the wave with the smallest wavelength changes depending on the frequency range. When below the characteristic frequency from Eq. (6), the slow wave has the smallest wavelength; when above, the shear wave does [17]. Since a large computational domain with an exceedingly fine mesh requires excessive computational cost, the domain length often must be smaller than ideal.

**MODEL VERIFICATION**

**Wave Speed Validation**

Since the commercial finite element solver used here employs a formulation with input parameters that differ from the Biot-Stoll formulation, it is good practice to ensure that the finite element models developed give results consistent with previous work. The first test performed compared both the real and imaginary components of the three wave speeds for various frequencies to the values found analytically by following the discussion in Chapter 1 of Stoll’s monograph [10]. The parameters used for the finite element models are derived from those given by Stoll and Kan [5]. Using these parameters, a simple model of a poroelastic domain with length \( 10 \lambda_{\text{max}} \) and uniform grid with element size \( \lambda_{\text{min}}/4 \) was created for frequencies varying from 30 Hz to 30 kHz. The figure below shows excellent agreement between both the real and imaginary components of the wave speeds.
A finite element model of two flat layers, a top layer of water and a bottom layer of poroelastic sediment, was created to calculate the reflection coefficient at 1 kHz for a series of grazing angles. Again, the computational domain was set to length $10 \lambda_{\text{max}}$ and a uniform mesh was generated with element size $\lambda_{\text{min}}/4$. The physics of these layers was coupled as described above and the incident field was set to a Gaussian tapered plane with amplitude 1 Pa and tapering parameter $g = L/4$. Since the amplitude of the incident field is 1 Pa, the amplitude of the scattered pressure field is equal to the reflection coefficient. The reflection coefficients found in this manner were compared to analytical calculations following Appendix B of Stoll’s monograph [10]. A plot showing this comparison is below.

While there is generally good agreement for grazing angles approaching normal incidence, there is quite a bit of disparity between the two curves for any grazing angle below approximately 50°. It is suspected that this discrepancy is due to issues with the implementation of the PMLs that could not be addressed by the time of submission. Once this issue is resolved and the reflection coefficients are in agreement, finite element models with rough poroelastic surfaces will be created to study scattering effects.
ACKNOWLEDGMENTS

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REFERENCES


