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2pUWa11. Implementing physical constraints for noise only normal mode shape estimation
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Many underwater acoustic tasks employ a vertical line array to sample a continuous wave pressure field. In the absence of strong local sources, the noise sampled by an array consists of both a spatially correlated and a spatially uncorrelated component. The correlated component is generally due to distant sources and can be described in terms of the modes of the waveguide. If the number of propagating modes is less than the array size, the diffuse noise lies within a lower dimensional subspace than the array data. The mode shapes defining this subspace can be estimated from noise measurements. Propagation physics constrain the mode shapes defining this subspace to be real, but the basis vectors obtained from a singular value decomposition of noise snapshots are generally complex. A phase rotation for each basis vector is required to rectify this. This work proposes a weighted average of the phase angles for each element that minimizes the variance of the rotation angle estimate. Simulations compare the proposed algorithm against prior approaches such as rotating each basis vector by the phase of the largest magnitude sample, rotating by the average of the phase, or taking the magnitude and ignoring the phase. [Supported by ONR.]

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Implementing Physical Constraints for Noise Only Normal Mode Shape Estimation

INTRODUCTION

Many signal processing tasks employ an array of sensors to sample a continuous wave field at a discrete set of spatial locations. This includes applications such as imaging, communication systems, remote sensing, and signal detection [1]. In the context of underwater acoustics this translates to line arrays, although this paper focuses on vertical line arrays specifically. Although a desired signal may or may not be present in any observation interval, the measurement at the array essentially always includes noise. The noise is modeled as a combination of a spatially correlated component and a spatially uncorrelated, or white, component. The uncorrelated component is generally modeling instrumentation and quantization noise within the array. The correlated component is generally due to causes outside of the array, such as distant sources and can be described in terms of the normal modes of the waveguide. If the number of propagating normal modes is less than the number of elements in the array, then the diffuse noise will lie in a lower dimensional subspace than the array data. However, this subspace is often not known in advance and may vary slowly over time as the environment changes physically. An accurate estimate of this subspace is an important element for many signal processing algorithms [1].

This research focuses on scenarios where there are a very limited number of observations containing the signal of interest, but assumes there is a large, easily accessible collection of “signal free” or “noise only” observations. Propagation physics often dictate that the received signals such as the desired signal and the diffuse noise field are constrained to fall in a lower dimensional subspace than the dimension of the full array. These same constraints apply both to the rare transient desired signals as well as the distant sources and diffuse excitation as they propagate to the array. Examples of this phenomenon are as diverse as mode propagation of electromagnetic waves in the ionosphere and also of acoustic mode propagation in underwater acoustics [2, 3]. Other waveguide propagation scenarios also would fit this model. There are several existing signal processing algorithms for estimating a basis for this subspace from data observations, including Principal Component Analysis (PCA), Discriminant Analysis, among others [4, 5, 6, 7]. However, these adaptive subspace methods generally do not account for constraints on the subspace structure that are implicit in the propagation physics. Depending on the array geometry and the nature of the waves in question, these constraints can be complicated or fairly straightforward.

The idea of estimating a subspace from data is not new, although prior work has been scarce. A paper by Neilsen and Westwood [8] is of particular interest. Part of the paper describes using the singular value decomposition (SVD) method to extract the vertical acoustic mode profiles from a given set of data. Their paper goes on to describe a method for separating data into a set of basis functions proportional to normal modes and singular values proportional to the mode eigenvalues. Unfortunately, the basis vectors that represent the shapes of the vertical modes are generally complex, while propagation physics dictate that these modes must be real or at least have common phase across the elements. To impose this constraint, the elements of the vector must all be rotated by a common angle, and then the real part taken of the vector. The challenge is finding the appropriate rotation angle. This research evaluates different rotation methods for imposing the common phase constraint and the methods’ effects on the quality of the mode shape estimation.

This paper is organized as follows: The following section covers the mathematics that lead up to the phase rotation methods. Next, we will introduce the phase rotation methods used in our investigation and discuss a pair of metrics that quantitatively measure how each methods affects the quality of the mode shape estimation. We then present the results to date and finally we present an analysis and discussion of the results.

BACKGROUND

For narrowband scenarios, a measurement of the signals at an N sensor array can be represented by an N element vector of complex phasors $y$, where each element of $y$ indicates the amplitude and phase of the narrowband signal at the corresponding array element. For the case of the physics constraining the signals to a linear manifold, this can be written as

$$ y = Ax + n, $$

(1)
where $A$ is an $N \times M$ matrix representing the $M$ dimensional linear subspace, $x$ is an $M$ element vector indicating the relative excitation of each dimension of $A$, and $n$ is an $N$ element vector of complex phasors representing the noise at the narrowband frequency. For example, in waveguide propagation the columns of $A$ would be the profiles of the shapes of the waveguide mode shapes at the sensor locations in the array. The vector $x$ would then contain the magnitude and phase for the excitation of each mode determined by the source location. Regardless of the source location and thus regardless of $x$, the observations are constrained to fall within the span of the matrix $A$. However, in many practical scenarios it is often not possible to solve for $A$ in advance without detailed knowledge of the environment.

As described earlier, the noise component $n$ in (1) is often a combination of several components. One of those components, the instrumentation noise, is spatially uncorrelated. The other component possesses spatial structure which appears as correlation across the array. The noise model is expressed as

$$n = n_c + n_w$$
$$= AD\phi + n_w,$$  \hspace{1cm} (2)

where $n_c$ is the correlated component and $n_w$ is the uncorrelated component. In the second line, the $M \times P$ matrix $D$ represents the collection of distant or diffuse sources, with the $P$ columns each representing a source, and the $P \times 1$ column vector $\phi = [e^{i\phi_1} ... e^{i\phi_P}]^T$ represents the random phase of the $P$ distant sources. This model makes it clear that the correlated component of the noise is also within the span of $A$.

In many scenarios, it is easy and convenient to obtain large collections of “noise only” data with no strong signals present, e.g., [8]. For this data, an observation $y$ contains only the $n_c$ and $n_w$ components. For these signal-free measurements, the covariance of the observed data $K_{yy}$ can be written as

$$K_{yy} = E\{yy^H\}$$
$$= E\{(n_c + n_w)(n_c + n_w)^H\}$$
$$= E\{AD\phi^H D^H A^H\} + E\{n_w n_w^H\}$$
$$= AK_{DD} A^H + \sigma_w^2 I,$$  \hspace{1cm} (3)

where $K_{DD}$ is the covariance matrix for the diffuse sources, $\sigma_w^2$ is the power in the instrumentation noise, and the superscript $(\quad)^H$ indicates the conjugate transpose (Hermitian) operator. This assumes that the random phases of the sources are independent and uniformly distributed, so that $E\{\phi^H \phi\} = I$. In many scenarios, $K_{DD}$ can reasonably be assumed to have a diagonal structure, simplifying the analysis. The goal of this research is to estimate the unknown but quasi-static subspace spanned by $A$ from a collection of signal-free observations $y_1, y_2, \ldots, y_L$. A first order approximation to an orthogonal basis for the span of $A$ is given by the eigenvectors of the sample covariance matrix

$$R_{yy} = \frac{1}{L} \sum_{l=1}^{L} y_l y_l^H.$$  \hspace{1cm} (4)

However, these eigenvectors do not incorporate constraints based on prior knowledge from the propagation physics and array geometry. One example of these constraints is when the array spans a waveguide such that the array axis is perpendicular to the direction of energy transport, such as in the vertical line array. In this case, the mode shapes (columns of $A$) are known to be real or at least in phase. If the observed vectors $y_l$ are complex, the eigenvectors of $K_{yy}$ will also be complex, and their elements are not necessarily in phase. This leads to the necessity for a phase rotation. This phase rotation is common for all elements of the vector $y$. Following the rotation, the real part of the vector $y$ is taken. If $y$ is a matrix then the phase rotation may be different for each column of $y$, but still common across all elements in a given column.

**PHASE ROTATION METHODS AND PERFORMANCE METRICS**

This investigation evaluates four rotation methods, three of which are currently used in practice. All of these methods perform a phase rotation unique to each column on the basis vector matrix. In the context of the acoustic waveguide this means each normal mode receives its own rotation independent of the other modes. For three of the four, this is accomplished by multiplying each column by $e^{-j\cdot\text{rot}a}$ where $\text{rot}a$ is the column’s phase rotation.
determined by the method. To determine the impact of each method on the quality of the mode profile estimations, two performance metrics were developed. The Correct Subspace Metric (CSM) describes how much of the estimate’s energy lies within the same subspace as the true modes. The Alignment Metric (AM) describes whether the estimate is properly aligned with the direction of the true modes. These are covered in more detail later.

**Phase Rotation Methods**

The Absolute Value Method (AVM) is the simplest of all the phase rotation methods. It only considers the magnitude of the complex mode profiles. The phase is ignored except to assign a sign to the magnitude. This method is, strictly speaking, not a phase rotation. Rather, it is a method that finds the signed magnitude, or amplitude, for each sample point of the complex mode profile. Sign information is determined by checking the numerical derivative of the phase for values above a threshold value.

The Strong Point Method (SPM) uses the phase of the array element with the largest magnitude as the phase rotation. This is an ad-hoc method that is used in practice. The premise behind this method can be seen in Figure 1 by imagining a polar point. If the point is close to the origin such that its magnitude is small, then small changes in the Cartesian plane (as a result of complex noise) result in large changes in phase. Similarly, if the point is far from the origin such that its magnitude is large, then small changes in the Cartesian plane result in small changes in phase.

![Figure 1](image.png)

**FIGURE 1.** This is a cartoon illustrating how the magnitude of an estimate can affect the variance of the phase estimate. The red line represents a polar point with a relatively small magnitude and the standard deviation of circular noise is represented by the red circle. Similarly, the blue line represents a polar point with a relatively large magnitude and the standard deviation of circular noise is represented by the blue circle. Note that both polar points have the same phase angle and the standard deviation of the noise is the same. The green lines represent the maximum deviation of the red polar point’s phase angle and the orange lines perform the same function for the blue polar point. It is important to notice that since the blue polar point has a larger magnitude, its deviation angle is much smaller than the red polar point’s. This means that an estimate with a large magnitude and the same complex additive noise has a smaller phase variance than an estimate with a small magnitude.

The third method currently used in practice is the Phase Averaging Method (PAM). This method averages the phase of the samples of the complex mode profile and uses the result as the phase rotation. Since the noise applied to the phase is circularly uniform the noise will disappear with an infinite number of samples. In practice, one does not have an infinite number of samples. However, since measurements of only noise are easily obtainable, one can obtain a very large number of samples. This method is complicated by the phase discontinuities that result from sign change. These must be accounted for, lest the mean be biased. These phase discontinuities are accounted for before averaging by checking the numerical derivative for values above a threshold.
Lastly, we introduce the Minimum Variance Method (MVM) which combines the best aspects of the SPM and PAM. MVM uses a weighted average of the phase of a complex mode profile. Different elements of the mode vector all give estimates of the same phase but with different variances. As discussed in the reasoning for the SPM, the variance is inversely proportional to the magnitude. Each sample of the phase is weighted by a value proportional to its relative magnitude and then averaged. This combines information about the shared phase with weights appropriate to the variance of each element. This expands upon PAM by assigning less importance to phases associated with small magnitudes and more importance to phases associated with larger magnitudes. Additionally, this method will collapse to PAM if the magnitude is uniform along the mode profile. This method also suffers from phase discontinuities due to sign changes. However, the phase discontinuities can be accounted for in the same manner as the PAM.

**Performance Metrics**

Two performance metrics have been developed to quantify the quality of an estimate. Both use the actual (non-corrupted) mode shapes along with the estimates of the mode shapes as the inputs. This is only possible if there is knowledge of the true mode shapes prior to estimating them. This research uses simulation data to evaluate the various phase rotation techniques, so the true modes are already known. If the matrix $A$ contains the actual mode shapes and $A_{est}$ contains the estimated mode shapes, then the singular value decomposition of these matrices produce $A = USV^T$ and $A_{est} = U_{est}S_{est}V_{est}^T$ respectively, where $S$ and $S_{est}$ are diagonal with the same dimensions as $A$ and $A_{est}$ and $U$, $V$, $U_{est}$, and $V_{est}$ are unitary matrices.

The Correct Subspace Metric (CSM) is given by $\|U^T U_{est}\|_F^2$. This metric gives a scalar value between zero and one that quantifies how much of the estimated mode profiles energy is in the same subspace as the actual modes. A value of one means all energy is in the same subspace as the actual modes and the estimates are in the correct subspace.

The Alignment Metric (AM) is given by $|A^T A_{est}|^2$. This metric gives a square matrix with dimensions of the number of modes in $A$ and $A_{est}$. This matrix describes how well the energy of the estimated modes align with the true modes by computing the magnitude squared of the inner product between each possible paring of actual and estimated modes. If the estimates are perfect, this results in a diagonal matrix. If $A$ and $A_{est}$ are have columns normalized by the respective L2-norm of the column, then the ideal AM reduces to the identity matrix. However, since samples of orthogonal functions are not necessarily orthogonal, even a perfectly aligned estimate will not produce a perfect identity matrix. However, all off diagonal terms will be much much smaller than the diagonal terms.

**RESULTS (TO DATE)**

This research uses simulated data to analyze the rotation methods. Since this research is conducted in the context of the underwater acoustic waveguide, several parameters must be specified. A frequency of 25 Hz was chosen in a water depth of 100 meters sampled by a 42 element array spanning the entire water column. To simplify the simulations and because this is a fairly shallow waveguide, the isovelocity sound speed profile was chosen with a sound speed of 1500m/s. From [9], this means that only three propagating modes are trapped by the waveguide. The mode shapes are fixed-free sinusoidal modes [10]. These modes are simulated using MATLAB and then correlated and uncorrelated noise of adjustable levels is applied. The signal to noise ratio (SNR) can be calculated by $\text{SNR} = 10 \cdot \log_{10} \left( \frac{\text{trace}(\text{Modes} \cdot \text{diag}(\sigma^2) \cdot \text{Modes}^H)}{\sum_{n=1}^{N} \sigma^2_n} \right)$, where Modes is an $N \times M$ matrix containing the uncorrupted modes along the columns, $\sigma_n^2$ is length $M$ vector containing the variances of the correlated noise, and $\sigma_0^2$ is the variance of the uncorrelated noise. Note that we have previously specified $N$ to be 42 and $M$ to be 3. An initial estimate of the mode shapes is calculated using the built-in eigenvector routine. This estimate is simultaneously used by all four rotation methods to estimate the needed phase rotation. These phase rotations are applied before the rotated estimates are forced to be real. Finally, all five estimates (the initial plus the four rotated estimates) are submitted to the metric.
functions along with the uncorrupted sinusoidal modes from the first stage. The most recent batch of simulations uses two parameters: number of snapshots and signal to noise ratio (SNR). Please note that no rotation (the initial estimate from the eigenvectors) has been shortened to NR.

**TABLE 1.** Below are the results for the Correct Subspace Metric (CSM) for no phase rotation (NR) and the four methods discussed in the previous section. Recall from the discussion on metrics that the best estimates will have a CSM near 1.

<table>
<thead>
<tr>
<th>CSM</th>
<th>NR</th>
<th>AVM</th>
<th>SPM</th>
<th>PAM</th>
<th>MVM</th>
</tr>
</thead>
<tbody>
<tr>
<td>42 Snapshots, 38 dB SNR</td>
<td>1.000</td>
<td>0.7277</td>
<td>1.000</td>
<td>0.9996</td>
<td>0.9961</td>
</tr>
<tr>
<td>126 Snapshots, 38 dB SNR</td>
<td>1.000</td>
<td>0.7316</td>
<td>1.000</td>
<td>1.000</td>
<td>0.9979</td>
</tr>
<tr>
<td>420 Snapshots, 38 dB SNR</td>
<td>1.000</td>
<td>0.7329</td>
<td>1.000</td>
<td>1.000</td>
<td>0.9985</td>
</tr>
<tr>
<td>42 Snapshots, -2 dB SNR</td>
<td>0.8998</td>
<td>0.6288</td>
<td>0.9115</td>
<td>0.7704</td>
<td>0.7497</td>
</tr>
<tr>
<td>126 Snapshots, -2 dB SNR</td>
<td>0.9730</td>
<td>0.6409</td>
<td>0.9734</td>
<td>0.8689</td>
<td>0.8515</td>
</tr>
<tr>
<td>420 Snapshots, -2 dB SNR</td>
<td>0.9923</td>
<td>0.6469</td>
<td>0.9923</td>
<td>0.9380</td>
<td>0.9270</td>
</tr>
</tbody>
</table>

**FIGURE 2.** Above are the results of Alignment Metric (AM) for the simulations with an SNR of 38 dB. They are separated by the number of snapshots of noisy data used. Since three modes were used, each method has a 3 by 3 matrix represented by a 3 by 3 block of colored squares. Please note that the color scale is logarithmic even though the shading is linear. Recall from the previous section that an ideal estimate would be the identity matrix. In terms of colors, the identity matrix would translate to three deep red squares along the diagonal surrounded by deep blue squares. The initial estimate and the four rotation estimates have been printed side by side for easy comparison across a trial with the same SNR and number of noise snapshots.
FIGURE 3. Above are the results of Alignment Metric (AM) for the simulations with an SNR of -2 dB. They are separated by the number of snapshots of noisy data used. Since three modes were used, each method has a 3 by 3 matrix represented by a 3 by 3 block of colored squares. Please note that the color scale is logarithmic even though the shading is linear. Recall from the previous section that an ideal estimate would be the identity matrix. In terms of colors, the identity matrix would translate to three deep red squares along the diagonal surrounded by deep blue squares. The initial estimate and the four rotation estimates have been printed side by side for easy comparison across a trial with the same SNR and number of noise snapshots.

ANALYSIS AND DISCUSSION

The results do not match our predictions. Since research is still ongoing, there may be a glitch or a violated mathematical assumption that has been overlooked. We had expected the MVM to perform best, followed by PAM, SPM, and finally AVM. As it stands, SPM generally provides the best estimate followed by PAM, MVM, and then AVM. Another curious result is that no rotation at all provides estimates that are generally comparable to SPM. Possible glitches in the simulation program could include a simple typographical error or improper correction of phase discontinuities.

Another possible is that one or more mathematical assumptions may have been violated during simulations. The most obvious is that there may not be enough noise snapshots. It is also possible that the simulated noise does not have the proper circular symmetry like Figure 1 implies. In addition, the phase noise for each element of the column should be uncorrelated. The phase rotation methods are linear operations on the estimated phase. It is possible that a non-linear phase correction is needed and the linear phase rotations are not sufficient. Another possibility is that the modes are supposed to be perfectly normal. However, as discussed previously, samples of orthogonal functions are not necessarily orthogonal. This means the columns of the mode shapes may not satisfy the orthogonality assumption. Finally, each column output from the SVD is a real single mode multiplied by a scalar complex exponential representing the phase. If the SVD is blending modes to form its orthogonal columns, no amount of phase estimation will help.

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