5aUW8. Efficient implementation of power-law attenuation of elastic waves in timedomain numerical simulations

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Numerical simulation of wave propagation in the time domain is easily parallelizable on high performance computing systems due to the spatially local nature of the governing equations. The disadvantage of working in the time domain arises when lossy media must be modeled which generally gives rise to convolution-type loss terms in the governing time-domain equations. Computation of these convolutions usually requires the storage of several solution fields at thousands of previous time steps. This requirement can be memory prohibitive in three-dimensions. In this talk we present a recursive convolution approach to computing lossy (power-law) elastic wave propagation that is an extension of the one-way, one-dimensional acoustic wave equation work done by Liebler (M. Liebler et al., J. Acoust. Soc. Am., 116, 2004) in order to handle multiple dimensions and shear waves. Convolutions are computed recursively by first using a least-squares technique to fit the kernel of the convolution with a series of decaying exponentials. We demonstrate how Graphical Processing Units (GPUs) can be used to obtain speed-up factors as high as 35 on a test computation of time-domain scattering from a highly resonant but lossy elastic cylinder. [Work sponsored by the Office of Naval Research]
INTRODUCTION

Direct calculation of elastic wave propagation in the time domain has advantages compared to frequency domain calculation such as ease of parallelization over large domains, visualization of wave resonances, and flexibility with implementation of radiation boundary conditions. A disadvantage occurs when wave attenuation, particularly power-law attenuation, is present. Although the no-loss equations are local at the spatial and temporal levels, attenuation requires the computation of a non-local (in-time) convolution term that requires knowledge of many previous solution states at a point (typically thousands). In two or three dimensions this requirement quickly becomes memory prohibitive.

A method to reduce the computational load of lossy time-domain calculations was put forth by Liebler et al. who worked with a one-dimensional fluid-only problem featuring one-way propagation. The approach consisted of approximating the decaying kernel of the convolution with a few decaying exponential terms. The exponential terms allow a recursive computation of the convolutions, similarly to how a perfectly matched layer can be approximating who worked with a one-dimensional solid elastic wave propagation in the time domain. Progress in particular the decaying kernel of the convolution with a few decaying exponential terms.

Experimental measurements of pulse propagation through lossy media indicate that power-law attenuation, a time-dependent form of the Kramers-Kröning spectral integral relations for the real and imaginary parts of the propagation wavenumber. For particular use in the case of power-law attenuation, a time-domain form of the Kramers-Kröning relations were derived by Szabo that make use of generalized function representations of the inverse Fourier transforms of the frequency domain loss functions. In a similar manner to the acoustic cases studied by Szabo and Liebler et al., we introduce the following elastic wave dispersive loss kernels

\[ L_p(t) = 2H(t) \left( \frac{1}{2\pi} \int \alpha_p(\omega)e^{-i\omega t} d\omega \right), \quad L_s(t) = 2H(t) \left( \frac{1}{2\pi} \int \alpha_s(\omega)e^{-i\omega t} d\omega \right). \]

The integrals can be expressed using generalized functions. The following equations closely follow the development in Liebler et al. The convolutions can be written in terms of the Riemann-Liouville definition of the fractional derivative

\[ L_p(t) * f(x,t) = h_{m,p} \Gamma(-y_p)D^y_p f(x,t), \quad L_s(t) * f(x,t) = h_{m,s} \Gamma(-y_s)D^y_s f(x,t), \]

where

**INCORPORATION OF ATTENUATION**

Power-Law Attenuation and Convolutions

Progressive sinusoidal compression (p) or shear (s) waves are observed to decay exponentially in space according to

\[ w(x + \Delta x) = w(x)e^{-\alpha\Delta x} \]  

where the loss function \( \alpha \) has the units of nepers/meter. The power-law frequency dependence is given by

\[ \alpha_p(\omega) = \alpha_p|\omega|^{\gamma}, \quad \alpha_s(\omega) = \alpha_s|\omega|^{\gamma}. \]  

Experimental measurements of pulse propagation through lossy media indicate that phase speed dispersion is always observed. The effect is formerly described by the Kramers-Kröning spectral integral relations for the real and imaginary parts of the propagation wavenumber. For particular use in the case of power-law attenuation, a time-domain form of the Kramers-Kröning relations were derived by Szabo that make use of generalized function representations of the inverse Fourier transforms of the frequency domain loss functions. In a similar manner to the acoustic cases studied by Szabo and Liebler et al., we introduce the following elastic wave dispersive loss kernels

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where

\[ h_{m,p} = \Gamma(-y_p), \quad h_{m,s} = \Gamma(-y_s). \]
D. Calvo and G. Canepa

**Formulations**

D. Calvo and G. Canepa found in Calvo et al.

\[ D^p f(x,t) = \frac{1}{\Gamma(-y_p)} \int_0^t f(x, \tau) \tau^{-y_p} d\tau, \quad D^s f(x,t) = \frac{1}{\Gamma(-y_s)} \int_0^t f(x, \tau) \tau^{-y_s} d\tau, \]  \(5\)

and

\[ h_{n,p} = \begin{cases} \left[ (-1)^{y_p} \right], & \text{for } y_p = 0, 2, 4... \\ \frac{2(y_p+1)[(-1)^v + 1]}{\pi(y_p+1)}, & y_p = 1, 3, 5... \\ \frac{2\Gamma(y_p+2)\cos\left(\frac{y_p+1}{2}\right)}{\pi(y_p+1)} , & \text{any real } y_p \end{cases} , \quad h_{n,s} = \text{same but with } y_s. \]  \(6\)

A discrete version of the convolution integral is given by the following

\[ D^p f(x,t) = \lim_{\Delta t \to 0} \frac{1}{(\Delta t)^y} \sum_{\ell=0}^\infty \frac{\Gamma(-y)}{\Gamma(-y)\Gamma(l+1)} f(x,t-l\Delta t). \]  \(7\)

The coefficients in the convolution can be made monotonically decaying by re-arranging the integral in terms of a second time derivative

\[ D^s f(x,t) = \lim_{\Delta t \to 0} \frac{1}{(\Delta t)^y} \sum_{\ell=0}^\infty \frac{\Gamma(-y+2)}{\Gamma(-y+2)\Gamma(l+1)} \partial^2 f(x,t-l\Delta t). \]  \(8\)

The coefficients of the second derivative, which we define as \(K(l,y)\), can be fit using a sequence of decaying exponentials

\[ K(l,y) = \tilde{K}(l,y) = \sum_{j=1}^N \tilde{a}_j \exp\left(-e_j l\right). \]  \(9\)

Canepa and Calvo \(^5\) found optimal fits of parameters \(a_i\) and \(e_i\) over a range of \(y\) by using a least-squares fit routine mexpfit.m.\(^9\) The general process of recursive convolution requires introducing a number of accumulator variables equal to the number of exponential terms. Specifics on advancing step-to-step in the time-domain algorithm can be found in Calvo et al or Liebler et al.\(^1,2\)

**Two-Dimensional Lossy Elastodynamic Equations**

We incorporate the dispersive loss operators into the two-dimensional elastodynamic equations in a stress relaxation form:

\[ \frac{\partial T_{xx}}{\partial t} + 2c_{\rho L} T_{xx} \ast \dot{\epsilon}_{xx} = \lambda \left( \frac{\partial \nu_x}{\partial x} + \frac{\partial \nu_z}{\partial z} \right) + 2\mu \frac{\partial \nu_x}{\partial x}, \]  \(10\)
\[
\frac{\partial T_{xz}}{\partial t} + 2c_{p,0}L_{p} \ast T_{xz} = \lambda \left( \frac{\partial v_{x}}{\partial x} + \frac{\partial v_{z}}{\partial z} \right) + 2\mu \frac{\partial v_{z}}{\partial z}, \tag{11}
\]

\[
\frac{\partial T_{zx}}{\partial t} + 2c_{s,0}L_{s} \ast T_{zx} = \mu \left( \frac{\partial v_{x}}{\partial x} + \frac{\partial v_{z}}{\partial z} \right), \tag{12}
\]

\[
\rho \frac{\partial v_{x}}{\partial t} = \frac{\partial T_{xz}}{\partial x} + \frac{\partial T_{zx}}{\partial z}, \tag{13}
\]

\[
\rho \frac{\partial v_{z}}{\partial t} = \frac{\partial T_{xz}}{\partial x} + \frac{\partial T_{zx}}{\partial z}. \tag{14}
\]

These equations govern two velocities \((v_{x}, v_{z})\), two normal stresses \((T_{xx}, T_{xz})\) and one shear stress \((T_{zx})\). The lossless \((L=0)\) equations can be found, for example, in Graff. The parameters \(\lambda\) and \(\mu\) are assumed to be lossless Lamé elastic constants which are related to the lossless compression and shear waves speeds according to

\[
c_{p,0} = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \quad c_{s,0} = \sqrt{\frac{\mu}{\rho}}. \tag{15}
\]

where \(\rho\) is the density of the material. An eigenmode analysis of the lossy equations \((10-14)\) shows the speeds of uncoupled compression and shear waves have the following form\(^3\)

\[
c_{p} = \sqrt{\frac{\lambda + 2\mu - 2i\rho c_{p,0}^{2} \frac{\hat{L}_{p}}{k_{p}}}{\rho}}, \quad c_{s} = \sqrt{\frac{\mu - 2i\rho c_{s,0}^{2} \frac{\hat{L}_{s}}{k_{s}}}{\rho}}, \tag{16}
\]

where \(k_{p} = \omega/c_{p,0}\) and \(k_{s} = \omega/c_{s,0}\) and the angular frequency is given by \(\omega\). The \(L\) functions can therefore be related to complex and generally frequency-dependent elastic moduli that can be measured by dynamic mechanical testing means (non-wave based methods).

**SCATTERING FROM AN UNDERWATER LOSSY ELASTIC CYLINDER**

The governing equations are solved using the elastodynamic finite integration technique (EFIT) which is a finite-volume time-domain solver.\(^7\) The computational grid is Cartesian with equal mesh resolution in the \(x\) and \(z\) directions. EFIT is closely allied with the Finite Difference Time Domain (FDTD) method\(^2\) in the special case of Cartesian grids. The approach to computing the backscattered spectrum is to calculate a backscattered time series from the cylinder as well as the time series at the cylinder center, but in the absence of the cylinder. The backscattering strength is defined as the ratio of Fourier transforms of these two time series.\(^7\) A receiving hydrophone was placed 25 cm in the backscatter direction from the cylinder center as shown in Figure 1. Water is assumed to have properties \(c_{p} = 1540\) m/s, \(c_{s} = 1,100\) m/s, \(\rho = 1000\) kg/m\(^3\). The composition of the 1 ft diameter (0.1524 m) solid cylinder is similar to hard rubber with \(c_{p} = 1100\) m/s, \(c_{s} = 350\) m/s, and \(\rho = 2000\) kg/m\(^3\). The compression and shear wave attenuations used are \(\alpha_{p} = 7.276E-8\) \([\omega]^{1.4}\) neper/m and \(\alpha_{s} = 1.455E-7[\omega]^{1.4}\) neper/m. An incident plane wave pulse is applied by creating a concentrated compressional body-force load along the yellow dotted vertical line in Figure 1. A 0.35 m thick recursive convolution PML layer (RCPML) for wideband absorption of boundary interacting signals is placed along the entire perimeter of the domain. This RCPML on the left absorbs both the left travelling radiated wave from the source and any sound backscattered from the cylinder. The source time history is a wide-band Ricker pulse centered at 3 kHz. The physical duration of the simulation is 0.142 sec which allows for fine frequency domain resolution of the backscattered spectrum.
The spatial discretization was calculated based on the slowest wave speed (shear in this case) and for the highest frequency of interest (6 kHz which is twice the center frequency of the Ricker pulse). A low-resolution grid of 15 points per wavelength was used, as was a high-resolution grid of 37 points per wavelength. The reference solution was computed using a finite-element method which was found to be in excellent agreement with the analytical separation of variables approach. The finite-element approach used frequency-dependent values of Young’s modulus and the Poisson ratio such that compression and shear wavenumbers had the required complex (lossy) parts for each frequency in the simulation. EFIT results are computed using 7 exponential fit terms for the convolution.

Results presented in Figure 2 show that as the grid resolution is increased, the backscattering strength computed by the EFIT code approaches the finite-element result. The cases with attenuation have been plotted on the same Figure and offset by a factor of 1.6 for convenience. Dampening of the peaks due to attenuation is apparent. It should be noted that we were not able to resolve the resonant peak at 4.3 kHz with the EFIT code. Both finite-element and separation of variables approaches were able to detect the presence of a resonant peak in those situations. The peak was not evident in lossy cases.

**NUMERICAL PERFORMANCE WITH GPU ACCELERATION**

Our efforts on accelerating the EFIT code by using Graphical Processing Units is discussed in Canepa & Calvo. Other efforts on acceleration and background on GPU architectures can be found in the literature. The model compiled for the CPU was run on a single core of an Intel® Xeon® CPU (X5472, 3 GHz clock). The model compiled for the GPU was tested on two different platforms (both with a single GPU). The first was the NVIDIA™ Tesla™ 2050, 448 cores, 3 GB RAM with error control. The second was the GeForce™ GTX 460, 336 cores, 1 GB RAM. Speed-up results are shown in Table 1. It should be noted that for a given mesh resolution, more time steps were required to complete the calculation. This was attributed to an instability associated with the second time-derivative in the loss terms which required reducing $\Delta t$ by a factor of 2.5. The high-end GPU delivers a speed up gain of roughly 30 for the no-loss and lossy cases for either mesh resolutions.

**DISCUSSION**

Although not reported here, spectral analysis of pulse propagation of bulk shear and compression waves were in good agreement with theory. Specifically, attenuation was verified to be as specified. Analysis of dispersion and comparison with Kramers-Krönig analytical predictions will be reported in separate work.

**FIGURE 1.** Schematic of EFIT simulation geometry. The plane wave source (dashed line in yellow) emits both left and right travelling waves. The left part of the RCPML absorbs both the left-going wave from the source and backscattering from the elastic cylinder. The domain is vertically symmetric.
FIGURE 2. Backscattering strength for the test cylinder defined as the Fourier transform of the backscattered signal at 0.25 m from cylinder center divided by the Fourier transform of the signal at cylinder center, but in absence of cylinder. The cases that include compression and shear wave attenuation are offset by 1.6 relative to the no-loss curves which naturally appear with more peaks due to strong resonances. The finite-element reference solution is approached as the mesh resolution increases.

TABLE 1. Computation times of the EFIT model of Figure 1 running on three different platforms, in two resolutions, and with and without attenuation. The total number of time steps for each run are also shown.

<table>
<thead>
<tr>
<th>CPU</th>
<th>GPU 1</th>
<th>GPU 2</th>
<th>Steps #</th>
<th>Attenuation</th>
<th>Resolution</th>
</tr>
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<tbody>
<tr>
<td>1300 s</td>
<td>49 s</td>
<td>78 s</td>
<td>108,240</td>
<td>No</td>
<td>Low</td>
</tr>
<tr>
<td>7670 s</td>
<td>278 s</td>
<td>465 s</td>
<td>270,600</td>
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<td>Low</td>
</tr>
<tr>
<td>22,590 s</td>
<td>634 s</td>
<td>1113 s</td>
<td>266,921</td>
<td>No</td>
<td>High</td>
</tr>
<tr>
<td>135,314 s</td>
<td>4231 s</td>
<td>6670 s</td>
<td>667,304</td>
<td>Yes</td>
<td>High</td>
</tr>
</tbody>
</table>

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REFERENCES