1aPAa4. Acoustic radiation force on a sphere without restriction to axisymmetric fields

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The analysis presented at the previous ASA meeting related to investigation of the acoustic radiation force on a sphere embedded in a soft elastic medium with shear modulus that is several orders of magnitude smaller than its bulk modulus. The acoustic field was assumed to be axisymmetric and the spherical scatterer to be located on the axis of the acoustic beam. When one of these conditions is violated the problem loses its symmetry. In this talk the acoustic radiation force is considered in the more general case of nonaxisymmetric fields. The calculation is performed in Lagrangian coordinates. All acoustic fields, incident as well as scattered, depend on all three spherical coordinates. The incident and scattered waves, which include both potential and solenoidal parts, are expanded with respect to spherical harmonics. An analytical expression for the acoustic radiation force derived in this investigation may contain as many spherical harmonics as needed. In limiting cases when the scatterer is in liquid and only two modes, monopole and dipole, remain in the scattered fields, the solution for the acoustic radiation force recovers the results reported by Gor'kov [Sov. Phys. Doklady 6, 773 (1962)]. [Work supported by NIH DK070618 and EB011603.]
INTRODUCTION

The approach to modeling acoustic radiation force on an elastic sphere of arbitrary size in tissue as described previously [1] for an axisymmetric acoustic field is generalized here to account for the radiation force on a sphere in tissue when the incident acoustic field is not axisymmetric. Analysis of the radiation force is performed in Lagrangian coordinates, which distinguishes it from previous analyses carried out in Eulerian coordinates and primarily for liquids. Two advantages of using Lagrangian coordinates may be noted: the surface of the sphere is fixed in the reference frame, and nonlinearity appears only in the stress tensor. Presented here are general results for acoustic radiation force that include both the potential and solenoidal components of the scattered fields. They are presented in a form that permits as many spherical harmonics as are needed to describe the field structure to be taken into account. There is no restriction on the size of the sphere. For a small sphere in liquid, classical results obtained by Gor'kov [2] and others are recovered.

GENERAL EQUATION FOR ACOUSTIC RADIATION FORCE FOR NONAXISYMMETRIC FIELD

Scattering of acoustic waves in tissue creates two fields, one that is purely potential like the incident waves, and another that is solenoidal because the medium surrounding the scatterer is elastic and characterized by a shear modulus $\mu$. Both scattered fields contribute to the acoustic radiation force and can be presented as

$$F_z = F_{z1} + F_{z2},$$  \hspace{1cm} (1)

where the component $F_{z1}$ of the radiation force is associated exclusively with the scattered potential field, whereas $F_{z2}$ includes contributions from the scattered solenoidal field.

Potential field contribution

According to Eq. (4) in the previous work [1], the potential component of the radiation force consists of two terms, which are expressed here in terms of the displacement potential $\phi$:

$$F_{z1} = Kk^2 \frac{R}{r} \int_S \left\{ \left( \frac{\partial}{\partial z} \left[ r \frac{\partial \phi}{\partial r} - \phi \right] \right) - \frac{R \cos^2 \theta}{2} \left( \left( \nabla \phi \right)^2 - k^2 \phi^2 \right) \right\} dS,$$  \hspace{1cm} (2)

where $K$ is the bulk modulus of the tissue, $k = \omega/c_l$ the wavenumber corresponding to the longitudinal sound speed $c_l$ in the tissue, and $R$ the radius of the spherical particle. Integration is performed over the surface $S$ of the sphere.

The displacement potential $\phi$ is expressed in spherical coordinates as

$$\phi = \frac{1}{2} \sum_n \sum_{m=-n}^n \left[ a_m^n L_n(kr) P_m^n(\cos \theta) e^{im\phi} e^{-i\omega t} + \text{c.c.} \right],$$  \hspace{1cm} (3)

where $a_m^n$ are amplitudes of the incident acoustic field, $L_n(kr)$ are expressed through spherical Bessel and Hankel functions $j_n(kr)$ and $h_n(kr)$, respectively, as

$$L_n(kr) = j_n(kr) + A_n h_n(kr),$$  \hspace{1cm} (4)

$A_n$ are scattering coefficients for the potential field, and $P_m^n(\cos \theta)$ are associated Legendre functions. Unlike in the case of an axisymmetric incident field, here the associated Legendre functions are required and a double summation is needed, one with respect to spherical modes $n$ and another with respect to the order of the Legendre functions $m$, which varies from $-n$ to $n$. 
Note that the scattering coefficients $A_n$ in Eq. (4) do not depend on $m$, a consequence of the spherical symmetry of the scatterer that simplifies the analysis.

The notation $\langle \cdots \rangle$ in Eq. (2) represents time averaging. The first time average follows from the Piola-Kirchhoff pseudostress tensor $\sigma_{kl}$, which is defined in terms of the elastic energy $\mathcal{E}$ according to

$$\sigma_{kl} = \frac{\partial \mathcal{E}}{\partial (\partial u_k / \partial x_l)}, \tag{5}$$

where $u_k = \partial \phi / \partial x_k$ are displacement components. The elastic energy density is expressed in the notation of Landau and Lifshitz [3], where the second-order elastic constants are the bulk modulus $K$ and shear modulus $\mu$, and the third-order elastic constants are $A$, $B$, and $C$. For soft tissue the ratio $\mu/K$ is in the range of $10^{-4} - 10^{-6}$, and measurements reported by Catheline et al. [4] reveal that $A = O(\mu)$ and $B = O(K)$. Under these conditions, with the further recognition that $K + B = O(\mu)$ [5], one obtains

$$\sigma_{kl} \approx -K \frac{\partial u_n}{\partial x_n} \frac{\partial u_l}{\partial x_k}. \tag{6}$$

The second time average in Eq. (2) takes into account the stress created by the tissue deformation that is caused by the volume force associated with the Piola-Kirchhoff stress tensor [1].

The final result for the potential part of the radiation force for a nonaxisymmetric incident acoustic field, or for a scatterer located off axis in an axisymmetric acoustic field, is

$$F_{z1} = i\pi K k^2 \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{(n + m + 1)(n + m)!}{(2n + 1)(2n + 3)(n - m)!} [A_n^* + A_{n+1}^* + 2A_n^* A_{n+1}^*] a_n^m a_{n+1}^m + \text{c.c.} \tag{7}$$

Equation (7) provides the $z$-component of the radiation force, where $z$ is the axis with respect to which the polar angle $\theta$ is defined. Other radiation force components are obtained by rotation of the coordinate system and use of Werner’s functions to transform the coefficients $a_n^m$.

**Solenoidal field contribution**

The scattered field contains a solenoidal component in spite of the fact that the incident field is acoustic, i.e., purely potential. The shear field contribution to the acoustic radiation force is written here using Eq. (6):

$$F_{z2} = -K \int_S \left\langle \frac{\partial u_n}{\partial x_n} \frac{\partial u_l}{\partial x_l} \right\rangle \frac{x_l}{R} dS = -\frac{k^2}{R} \int_S \left\langle \frac{\partial u_l}{\partial z} \right\rangle x_l dS. \tag{8}$$

Components of the displacement in the solenoidal field are determined from the equation for the displacement vector $u(r, \theta, \phi)$,

$$u(r, \theta, \phi, t) = \frac{1}{2} \nabla \times \nabla \times \left[ \Pi(r, \theta, \phi) r \right] e^{-i\omega t} + \text{c.c.}, \tag{9}$$

where $\mathbf{r}$ is a position vector and

$$\Pi(r, \theta, \phi) = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} B_n a_n^m \Pi_n^m (r, \theta, \phi). \tag{10}$$

Here, $B_n$ are shear wave scattering coefficients, and

$$\Pi_n^m (r, \theta, \phi) = h_n (kr) P_n^m (\cos \theta) e^{im\phi}, \tag{11}$$

where $\kappa = \omega/c_\tau$ is a wavenumber, and $c_\tau$ is the shear wave propagation speed. The function $\Pi_n^m (r, \theta, \phi)$ satisfies the Helmholtz equation for shear waves,

$$\nabla^2 \Pi_n^m + \kappa^2 \Pi_n^m = 0. \tag{12}$$
The solenoidal field is thus characterized by the function $\Pi(r, \theta, \phi)$. Note that the scattering coefficients $B_n$ of the solenoidal field for the nonaxisymmetric acoustic field are independent of $m$.

The final result for the shear wave contribution to the acoustic radiation force due to the solenoidal field for acoustic wave scattering by a spherical obstacle is

$$F_{z2} = -\pi K k^2 r \sum_{n=1}^{\infty} \frac{n(n+2)}{(2n+1)(2n+3)} \sum_{m=-n}^{n} \frac{(n+m+1)!}{(n-m)!} \times \left\{ a_n^m r^{n+1} B_{n+1} j_n^*(kR) + A_n^m h_n^*(kR) h_n(kR) \right\} \right|_{r=R} + c.c. \right.$$  \hspace{1cm} (13)

Equations (7) and (13) reveal that the acoustic radiation force depends on the scattering coefficients $A_n$ for the potential field and $B_n$ for the solenoidal field, as well as on the incident acoustic field amplitude coefficients $a_n^m$. The scattering coefficients $A_n$ and $B_n$ are determined by solving equations for the boundary conditions on the particle surface, i.e., at $r=R$, whereby the initial field is purely potential while the scattered field consists of potential and solenoidal components, and the transmitted field (interior of the sphere) also contains both potential and solenoidal components.

The coefficients $a_n^m$ arise from a spherical harmonic expansion of the incident field. From a practical point of view, it is better to express $a_n^m$ through an incident acoustic field that may be known through measurements or numerical estimation. As an example, we consider the case where the incident field is formed by only the monopole and dipole modes. In this case $F_{z2} = 0$, but this does not mean that the scattered field is independent of the elasticity of the medium. For small $kR$ it has been shown [1] that the scattering coefficients $A_0$ and $A_1$ are given by

$$A_0 = \frac{i}{3} (kR)^3 \left( \frac{K}{K_p} - 1 \right) \left[ 1 + \frac{4 \mu}{3 K_p} - \frac{i}{3} (kR)^3 \left( \frac{K}{K_p} - 1 \right) \right]^{-1},$$

$$A_1 = \frac{i}{3} (kR)^3 \left( \frac{\rho - \rho_p}{\rho + 2 \rho_p} \right) \left[ 1 + \frac{i}{3} (kR)^3 \left( \frac{\rho - \rho_p}{\rho + 2 \rho_p} \right) \right]^{-1},$$  \hspace{1cm} (14)

where $K = \lambda + \frac{2}{3} \mu$ and $K_p = \lambda_p + \frac{2}{3} \mu_p$ are the bulk moduli of the medium and the particle, respectively. To evaluate the effect of finite shear modulus we write Eqs. (14) for monopole scattering by a gas bubble at frequencies well below the natural frequency of the bubble pulsation:

$$A_0 \approx \frac{i}{3} (kR)^3 \frac{K}{\gamma P_0} \left( 1 + \frac{4 \mu}{3 \gamma P_0} \right)^{-1},$$  \hspace{1cm} (15)

where $K_p = \gamma P_0$, $\gamma$ is the ratio of specific heats for the gas, and $P_0$ is atmospheric pressure.

Previous estimations [1] demonstrate that $F_{z1}$ in tissue may be about 40% less than the value predicted for a gas bubble in liquid.

**Two-mode analysis and comparison with Gor’kov results**

**Parameters of incident acoustic field**

The incident acoustic wave field is expanded in spherical harmonics:

$$\psi(r, \theta, \phi) = a_0 j_0(kr) + \sum_{m=-1}^{1} a_1^m P_1^m(\cos \theta) e^{im\phi} j_1(kr) + \sum_{m=-2}^{2} a_2^m P_2^m(\cos \theta) e^{im\phi} j_2(kr) + \cdots,$$  \hspace{1cm} (16)

where $\psi$ is the complex displacement potential amplitude, which is related to the real displacement potential by

$$\varphi = \frac{1}{2} \left( \psi e^{-iat} + \psi^* e^{iat} \right),$$  \hspace{1cm} (17)
and \( j_n(\xi) \) are the standard spherical Bessel functions. [6] However, here we use modified associated Legendre functions \( \tilde{P}_n^m \), which are related to the standard associated Legendre functions by

\[
\tilde{P}_n^m = \begin{cases} 
P_n^m & \text{for } m > 0, \\
(-1)^m P_n^{|m|} & \text{for } m < 0.
\end{cases}
\] (18)

Use of the modified associated Legendre functions simplifies the calculations.

We now expand \( \psi(x, y, z) \) in a Taylor series about the origin through second order in the coordinates:

\[
\psi(x, y, z) = \psi_0 + \left( \psi_x x + \psi_y y + \psi_z z \right) + \frac{1}{2} \left( \psi_{xx} x^2 + \psi_{yy} y^2 + \psi_{zz} z^2 + 2 \psi_{xy} xy + 2 \psi_{xz} xz + 2 \psi_{yz} yz \right),
\] (19)

where \( \psi_0 = \psi(0, 0, 0) \), and subscripts denote partial derivatives with respect to the corresponding coordinates. Substitution of the relations

\[
x = r \sin \theta \cos \phi = \frac{r}{2} \sin \theta \left( e^{i \phi} + e^{-i \phi} \right), \quad y = r \sin \theta \sin \phi = \frac{r}{2i} \sin \theta \left( e^{i \phi} - e^{-i \phi} \right), \quad z = r \cos \theta
\] (20)

into Eq. (19) yields

\[
\psi = \psi_0 + \frac{r}{2} \psi_x \sin \theta \left( e^{i \phi} + e^{-i \phi} \right) + \frac{r}{2i} \psi_y \sin \theta \left( e^{i \phi} - e^{-i \phi} \right) + r \psi_z \cos \theta + \frac{r^2}{8} \psi_{xx} \sin^2 \theta \left( e^{2i \phi} + e^{-2i \phi} + 2 \right) - \frac{r^2}{8} \psi_{yy} \sin^2 \theta \left( e^{2i \phi} + e^{-2i \phi} - 2 \right) + \frac{r^2}{2} \psi_{zz} \cos^2 \theta + \frac{r^2}{2} \psi_{xy} \sin \theta \cos \theta \left( e^{i \phi} + e^{-i \phi} \right) + \frac{r^2}{2i} \psi_{xz} \sin \theta \cos \theta \left( e^{i \phi} - e^{-i \phi} \right) + \frac{r^2}{4i} \psi_{yz} \sin^2 \theta \left( e^{2i \phi} - e^{-2i \phi} \right).
\] (21)

We likewise expand the Bessel functions through terms of second order in the argument for a small scatterer of radius \( R \), such that \( \xi = kr = kR \ll 1 \):

\[
j_0(\xi) \approx 1 - \frac{\xi^2}{6}, \quad j_1(\xi) \approx \frac{\xi}{3}, \quad j_2(\xi) \approx \frac{\xi^2}{15}.
\] (22)

Since only the Bessel functions \( j_0, j_1 \) and \( j_2 \) are employed (because all others are of higher order than \( \xi^2 \)), only a limited number of the functions \( \tilde{P}_n^m \) are required, which can be calculated using the relation [6]

\[
P_n^m(x) = (-1)^m (1 - x^2)^{m/2} \frac{d^m P_n(x)}{dx^m}, \quad \text{where} \quad x = \cos \theta.
\] (23)

Comparison of Eqs. (16) and (21) yields

\[
\begin{align*}
a_0 = & \psi_0 = \psi(0, 0, 0), \\
a_1 = & \frac{3}{2k} (\psi_x + i \psi_y), \\
a_2 = & \frac{3}{k} \psi_z, \\
a_3 = & -\frac{3}{2k} (\psi_x - i \psi_y), \\
a_4 = & \frac{5}{8k^2} (\psi_{xx} - \psi_{yy} + 2i \psi_{xy}), \\
a_5 = & \frac{5}{2k^2} (\psi_{xz} + i \psi_{yz}), \\
a_6 = & \frac{5}{2k^2} (3 \psi_{zz} + k^2 \psi_0), \\
a_7 = & -\frac{5}{2k^2} (\psi_{xz} - i \psi_{yz}), \\
a_8 = & \frac{5}{8k^2} (\psi_{xx} - \psi_{yy} - 2i \psi_{xy}).
\end{align*}
\] (24)
In terms of the coefficients in the spherical harmonic expansion employing the modified associated Legendre functions, Eq. (7) becomes

$$F_{z1} = i\pi Kk \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left( \frac{A_n^* + A_{n+1}^* + 2A_n^*A_{n+1}}{(2n+1)(2n+3)} \right) \frac{\sum_{n=-n}^{n} (n+1+|m|)!}{(n-|m|)!} A_m^*a_n^* + \text{c.c.}$$  \hspace{1cm} (25)

For the two-mode analysis only \( A_0 \) and \( A_1 \) remain in Eq. (25), the product \( A_0^*A_1 \) is neglected, and substitution of Eqs. (24) into Eq. (25) yields

$$F_{z} \approx \pi Kk \left[ iA_0^*\psi_0^*\psi_2 + \frac{3iA_1^*}{k^2} \left( \frac{\partial \psi_x}{\partial z} \psi_x^* + \frac{\partial \psi_y}{\partial z} \psi_y^* + \frac{\partial \psi_z}{\partial z} \psi_z^* \right) \right] + \text{c.c.}$$

**Comparison with Gor'kov results**

In the case where \( A_0 \) and \( A_1 \) are pure imaginary, i.e., \( A_0^* = -A_0, A_1^* = -A_1 \), Eq. (26) becomes

$$F_{z1} \approx -i\pi Kk \frac{\partial}{\partial z} \left[ A_0\psi\psi^* + \frac{3A_1}{k^2}(\psi_x\psi_x^* + \psi_y\psi_y^* + \psi_z\psi_z^*) \right].$$  \hspace{1cm} (27)

Now take into account that

$$p = \rho \omega^2 \psi, \quad v = -i\omega \nabla \psi, \quad \psi^* = \frac{2}{\rho \omega^2} p_{ac}, \quad \psi_x^* = \frac{2}{\omega^2} v_{ac,x},$$

where \( p_{ac}(t) \) and \( v_{ac,x}(t) \) are the real acoustic field variables, related to their complex amplitudes, following time averaging (designated here by overbars for the purpose of comparing with Gor'kov), by

$$\overline{\frac{p_{ac}}{\rho c}} = \frac{1}{4} \left( \frac{pe^{-iat} + \text{c.c.}}{\rho c} \right), \quad \overline{v_{ac,x}} = \frac{1}{4} \left( \frac{v_{ac}e^{-iat} + \text{c.c.}}{\rho c} \right).$$  \hspace{1cm} (29)

Finally, introducing the functions \( f_1 \) and \( f_2 \) used by Gor'kov,

$$f_1 = 1 - \frac{K}{K_p}, \quad f_2 = \frac{2(\rho_p - \rho)}{2\rho_p + \rho}, \quad K = \rho c^2,$$

which are related to the scattering coefficients \( A_0 \) and \( A_1 \) by

$$iA_0 = \frac{1}{3}(kR)^3 f_1, \quad iA_1 = -\frac{1}{6}(kR)^3 f_2,$$

one obtains

$$F_{z1} = -2\pi \rho R^2 \frac{\partial}{\partial z} \left[ \frac{\overline{p_{ac}}}{3(\rho c^2)} f_1 - \frac{\overline{v_{ac,x}}}{2} f_2 \right].$$  \hspace{1cm} (32)

Equation (32) is exactly the same as Eq. (12) of Gor'kov [2]. It should be noted that the Gor'kov formula is valid only in the low-frequency limit where the scattering coefficients are pure imaginary, and when they are defined only by the compressibility and density of the particle and surrounding liquid.

In the case where \( A_0 \) and \( A_1 \) are not pure imaginary, for example, when the scatterer is a liquid drop with high viscosity, or for a gas bubble when the frequency of the incident acoustic field is comparable to the resonance frequency of the bubble, then Eq. (32) is no longer valid, and Eq. (26) should be used to estimate the acoustic radiation force instead of Eq. (27). In this case we have

$$\psi_0 = \frac{p}{\rho \omega^2}, \quad \psi_x = \frac{i v_x}{\omega}, \quad \psi_y = \frac{i v_y}{\omega}, \quad \psi_z = \frac{i v_z}{\omega},$$

and the acoustic radiation force can be presented as

$$F_{z1} \approx -2\pi Kk \left\{ \frac{1}{(\rho c^2)^2} \text{Re} \left[ iA_0 \frac{\partial p^*}{\partial z} \right] + \frac{3}{k^2 \omega^2} \text{Re} \left[ iA_1 \left( \frac{\partial v_x^*}{\partial z} + v_y \frac{\partial v_x^*}{\partial z} + v_z \frac{\partial v_x^*}{\partial z} \right) \right] \right\}. $$ \hspace{1cm} (34)
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