1aPAb1. Perturbation analysis of flow about spherically pulsating bubble at the velocity node of a standing wave

Mohammad K. AlHamli, Alexey Y. Rednikov and Satwindar S. Sadhal*

*Corresponding author's address: Aerospace & Mechanical Engineering, University of Southern California, Los Angeles, CA 90089-1453, sadhal@usc.edu

An analysis using the singular perturbation method for a radially pulsating gas bubble at the velocity node of a standing wave was conducted with $\varepsilon = U_0/(a\omega) \ll 1$ as a small parameter and $\omega a^2/\nu \gg 1$ as a large parameter. Here, $a$, $U_0$, $\omega$ and $\nu$ are length scale, velocity scale, frequency and kinematic viscosity, respectively. While the mean oscillatory flow around the gas bubble has no net time-averaged flow component, viscous steady streaming arises due to the nonlinearity of the flow dynamics. However, with bubble surface being considered shear-free, the vorticity generation in the system is quite weak as compared with what would result from a solid boundary. Not surprisingly, the steady streaming is also weak. As already known, the steady streaming would not arise with purely radial pulsations of a bubble in an otherwise quiescent liquid. For the case of a non-pulsating bubble at the node, streaming is seen at $O(\varepsilon^2)$. However, as seen with the case of a radially pulsating bubble at the velocity antinode, interaction of two oscillatory fields creates streaming at lower order. The phase difference between radial and lateral oscillations was found to play a significant role in both the streaming direction and intensity.

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1- Problem Statement:
The acoustic streaming due to a gas bubble in a liquid medium subjected to a high-frequency standing wave has been investigated. We consider the situation when the bubble locates itself at the velocity node of the standing wave. This can happen when the particle phase has higher compressibility than the external phase (Zhao et al. 1999). The bubble will undergo both lateral and radial oscillations of the same magnitude, i.e. the angular frequency, \( \omega \), will be the same for both oscillation modes. However there could be a phase shift, \( \phi \), between the two modes of oscillations. Spherical polar coordinates with the origin at the center of the bubble have been adopted. The z-axis passes through the center of the sphere and points along the direction of vibration (see Figure 1).

The following dimensionless parameters will be used:

\[
\begin{align*}
Re &= \frac{U_\infty a}{\nu}; & M^2 &= \frac{i \omega a^2}{\nu}; & \varepsilon &= \frac{U_\infty}{\omega a} = \frac{Re}{|M|^2} \ll 1; & \varepsilon' &= \frac{U'}{\omega a} \ll 1
\end{align*}
\]

where \( U_\infty \), \( U' \), \( \nu \), \( a \), \( Re \), and \( M \) are the characteristic velocity for the lateral oscillation, the characteristic velocity of the radial oscillation, the kinematic viscosity of the medium, the radius of the bubble, the Reynolds number, and the frequency parameter, respectively. The parameters \( \varepsilon \) and \( \varepsilon' \) are the ratios of the lateral displacement amplitude and the radial displacement amplitude to the radius of the bubble, respectively. Here the assumption will be a high frequency standing wave so that \( |M|^2 \gg 1 \).

For a standing wave the velocity may expressed as

\[
u_z = U_\infty \cos [k(z + Z_0)] e^{i \omega t}
\]

where \( k = \omega/c \) is the wavenumber. The local velocity in the neighborhood of the velocity node, i.e. \( Z_0 = \frac{\pi}{2} \), is

\[
u_z = -U_\infty \sin (kz) e^{i \omega t} = -U_\infty \left( k z - \frac{k^2 z^3}{6} + \ldots \right) e^{i \omega t}.
\]

For a small particle at the velocity node we take only the first term of the expansion,

\[
u_z = -U_\infty k z e^{i \omega t}.
\]

Away from the bubble the flow will be irrotational; therefore a potential function can be defined, i.e.,

\[\mathbf{u} = \nabla \varphi.\]

The far-field potential function will be

\[
\varphi_\infty = \frac{U_\infty}{k} \cos (k z) e^{i \omega t} = \frac{U_\infty}{k} \left( 1 - \frac{k^2 r^2 \cos^2 \theta}{2} + \ldots \right) e^{i \omega t}.
\]

1.1 Equations of Motion and Dimensionless Scaling
The flow parameters are scaled as follows:

\[
\mathbf{u} = \frac{U'}{U_\infty}, \quad \psi = \frac{\psi'}{U_\infty}, \quad \mathbf{x} = \frac{x'}{a}, \quad \varphi = \frac{\varphi'}{U_\infty}, \quad \tau = \omega t, \quad p = \frac{p'}{\rho_c U_\infty \omega a},
\]

\[
\rho = \frac{\rho_c}{\rho_c U_\infty \omega a}, \quad \text{and} \quad \nabla = a \nabla'.
\]
where the asterisks denote the dimensioned quantities, $\psi$ is the stream function, $a$ is the bubble nominal radius, $t$ is time, $\rho_c$ is the constant medium density, and $p$ and $\rho$ are the acoustic pressure and density. Using the adiabatic relation, $\rho = p/c^2$, the dimensionless parameter $\rho$ and $\rho$ are equal.

The dimensionless equations of continuity and motion are

$$
(ka)^2 \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{u} + \varepsilon (ka)^2 \nabla \cdot (\rho \mathbf{u}) = 0
$$

(8)

$$
(1 + \varepsilon (ka)^2) \frac{\partial \mathbf{u}}{\partial t} + \varepsilon (1 + \rho \varepsilon (ka)^2) \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \frac{1}{|\mathbf{M}|^2} \nabla^2 \mathbf{u} = 0
$$

(9)

There are two boundary conditions at the surface of the bubble. The first one is the free surface or no shear stress:

$$
\tau_{r\theta} = 0 \quad \text{at} \quad r = R
$$

(10)

where $R(t) = a - \omega^2 \sin(\omega t)$, and in dimensionless form with radial phase shift $\varphi$ included,

$$
R(\tau) = 1 - i \varepsilon e^{i(\varphi + \varphi)}
$$

(11)

The free shear surface condition may also take the form,

$$
\frac{\partial}{\partial t} \left( \frac{u_0}{r} \right) = 0 \quad \text{at} \quad r = R
$$

(12)

The other boundary conditions at the surface of the bubble corresponds to the radial oscillation of the pulsating sphere,

$$
u_r = \frac{dR}{d\tau} = \varepsilon e^{i(\varphi + \varphi)} \quad \text{at} \quad r = R(\tau)$$

(13)

At the far field, i.e. $r \to \infty$, the velocity, the stream function, and the potential function are given by

$$
u_r = -kar \cos \theta e^{i\tau},
$$

(14)

$$\psi = -kar^2 \varepsilon \cos(\theta) \sin^2(\theta) e^{i\tau},
$$

(15)

and,

$$
\varphi_{oo} = \left[ \frac{1}{ka} - \frac{kar^2}{6} - \frac{kar^2}{3} P_2(\bar{\mu}) \right] e^{i\tau}
$$

(16)

where $P_2(\bar{\mu}) = \frac{1}{2}(3\bar{\mu} - 1)$, is the Legendre polynomial, and $\bar{\mu} = \cos \theta$.

### 2- Solution:

The perturbation method is applied to expand the velocity, acoustic pressure and density,

$$
\mathbf{u} = u_0 + \varepsilon \mathbf{u}_1 + O(\varepsilon^2),
$$

(17)

$$p = p_0 + \varepsilon p_1 + O(\varepsilon^2),
$$

(18)

and,

$$\rho = \rho_0 + \varepsilon \rho_1 + O(\varepsilon^2).
$$

(19)

2.1- The Leading Order Solution

Inserting (17), (18), and (19) into the momentum equation (9), the leading order velocity becomes

$$
\frac{\partial \mathbf{u}_0}{\partial \tau} = -\nabla p_0,
$$

(20)

which indicates that the leading order solution is irrotational. Therefore, the leading velocity $\mathbf{u}_0$ can be represented as a potential function

$$
\mathbf{u}_0 = \nabla \varphi_0
$$

(21)

Equation (20) becomes

$$
\frac{\partial \varphi_0}{\partial \tau} = -p_0
$$

(22)

and since the potential function in the far-field is known, the pressure and density can be found by applying equation (22) to $\varphi_{oo}$ in equation (16), i.e.,

$$p_{oo} = \rho_{oo} = -i \left[ \frac{1}{ka} - \frac{kar^2}{6} - \frac{kar^2}{3} P_2(\bar{\mu}) \right] e^{i\tau}.
$$

(23)

Applying the perturbation expansion to the continuity equation (8), the leading order gives
\[(ka)^2 \frac{\partial \rho_0}{\partial \tau} + \nabla \cdot \mathbf{u}_0 = 0. \tag{24}\]

With small \((ka)\), only the term \(-i \frac{1}{ka} e^{i\tau}\) will be needed from equation (23). Then the continuity equation becomes

\[ka^2 \frac{\partial \rho_0}{\partial \tau} + \nabla \cdot \mathbf{u}_0 = 0. \tag{25}\]

Writing equation (25) in terms of potential function, \(\varphi_0\),

\[\nabla^2 \varphi_0 + ka^2 e^{i\tau} = 0 \tag{26}\]

Solving the above equation and using the far field condition equation (16), and the normal velocity at the surface of the bubble i.e. \(u_\eta = \frac{\partial \varphi_0}{\partial r} = e^{i(\tau + \phi)}\) at \(r=R\), the solution takes the form,

\[\varphi_0 = \left[1 - \frac{ka}{3} \left(\frac{r^2}{2} + \frac{R^3}{r}\right) - \frac{ka}{3} \left(\frac{r^2 + 2 R^5}{3 r^3}\right) P_2(\tilde{\mu})\right] e^{i\tau} - \frac{R^2}{r} e^{i(\tau + \phi)}, \tag{27}\]

and the leading order acoustic pressure and density are

\[p_0 = \rho_0 = -i \left[1 - \frac{ka}{3} \left(\frac{r^2}{2} + \frac{R^3}{r}\right) - \frac{ka}{3} \left(\frac{r^2 + 2 R^5}{3 r^3}\right) P_2(\tilde{\mu})\right] e^{i\tau} + i \frac{R^2}{r} e^{i(\tau + \phi)}. \tag{28}\]

\[2.1.1 \text{- The Boundary Layer}\]

In the boundary layer, the velocity can be expressed as

\[u^b = u^b_\tau \hat{\tau} + u^b_\theta \hat{\theta} \tag{29}\]

where the superscript “b” indicates the boundary layer, \(u^b_\tau\) the normal velocity, and \(u^b_\theta\) the tangential velocity. When \(|M|^2 > 1\) the vorticity will be confined to the Stokes layer that has an order of \(|M|^{-1}\) i.e. \((\delta \sim 1/|M|)\) (Riley, 1966). Therefore the normal velocity inside the Stokes layer can be scaled to

\[u^b_\eta = \frac{|M|}{\sqrt{2}} u^b_\tau, \tag{30}\]

and the inner variable within the boundary layer

\[\eta = \frac{|M|}{\sqrt{2}} (r - R) \text{ so that when } r = R, \quad \eta = 0. \tag{31}\]

Next, we expand the velocity, acoustic pressure, and acoustic density in the boundary layer to find the perturbation solution, i.e.,

\[u^b = u^b_\tau + \epsilon u^b_\theta + O(\epsilon^2) , \tag{32}\]

\[p^b = p^b_0 + \epsilon p^b_\theta + O(\epsilon^2) , \tag{33}\]

\[\rho^b = \rho^b_0 + \epsilon \rho^b_\theta + O(\epsilon^2) . \tag{34}\]

Inserting equations (32), (33), and (34) into the momentum equation (9) results in

\[\frac{\partial p^b_\theta}{\partial \eta} = 0 \tag{35}\]

\[\frac{\partial u^b_\theta}{\partial \tau} = \frac{1}{R} \frac{\partial p^b_\theta}{\partial \tilde{\mu}} + \frac{1}{2} \frac{\partial^2 u^b_\theta}{\partial \eta^2}. \tag{36}\]

From equation (35), a conclusion that can be made is that the leading order pressure in the Stokes layer is a function of \(\tau\) and \(\theta\) only. Then,

\[p^b_\theta = \rho^b_\theta = p^b_0|_{r=R} = -i \left[1 - \frac{ka}{2} \frac{R^2}{r^2} - \frac{5}{9} kaR^2 P_2(\tilde{\mu})\right] e^{i\tau} + i \epsilon R e^{i(\tau + \phi)}. \tag{37}\]

From the above equation (37) and using the free surface boundary condition, given by equation (12), we have for the leading order

\[\frac{\partial u^b_\theta}{\partial \eta}|_{\eta=0} = \frac{\sqrt{2}}{|M|R} u^b_\eta|_{\eta=0}. \tag{38}\]

In equation (38) the term that was multiplied by \(\frac{1}{|M|}\) will not be omitted because \(\frac{\partial u^b_\theta}{\partial \eta}|_{\eta=0}\) is really small and comparable with \(\frac{\sqrt{2}}{|M|R} u^b_\eta|_{\eta=0}\). The solution for the leading order tangential velocity was found to be

\[u^b_\eta = \frac{5}{3} kaR\tilde{\mu}\sqrt{1 - \tilde{\mu}^2} \left[1 - \frac{\sqrt{2} e^{(1+i)\eta}}{\sqrt{2} + (1 + i)|M|R}\right] e^{i\tau}. \tag{39}\]
Again, inserting equations (32), (33), and (34) into the continuity equation (8), results in the leading order normal velocity equation,
\[ (ka)^2 \frac{\partial \rho_b^b}{\partial \tau} + \nabla \cdot u_b^b - \frac{\sqrt{1 - \bar{\mu}^2}}{R} \frac{\partial u_b^b}{\partial \mu} + \frac{\bar{\mu} u_b^b}{R\sqrt{1 - \bar{\mu}^2}} = 0, \tag{40} \]
With boundary condition for the normal velocity, \( u_{b0}|_{n=0} = \frac{|M|}{\sqrt{2}} \epsilon^\prime e^{i(\tau + \phi)} \), the solution for the leading order normal velocity was found to be
\[ u_{b0} = \left\{ -ka \eta + \frac{10}{3} ka \left[ -\eta + \frac{(1 + i)(1 - e^{-(1+i)\eta})}{2 + \sqrt{2}(1+i)|M|R} P_2(\bar{\mu}) \right] e^{i\tau} + \frac{|M|}{\sqrt{2}} \epsilon^\prime e^{i(\tau + \phi)} . \tag{41} \]

2.2 The First Order Solution, \( O(\epsilon) \):

For the first order solution, the interest is in acoustic streaming which is independent of time. Therefore we consider only the steady state solution. The continuity equation inside the boundary layer in the order of \( \epsilon \) is
\[ (ka)^2 \frac{\partial \rho_b^b}{\partial \tau} + \nabla \cdot u_b^b + (ka)^2 \nabla \cdot (\rho_b^b u_b^b) = 0 \tag{42} \]
For steady streaming, taking the time average of the above equation, we obtain
\[ \nabla \cdot u_b^b + ((ka)^2 \nabla \cdot (\rho_b^b u_b^b)) = 0 \tag{43} \]
where \( ((ka)^2 \nabla \cdot (\rho_b^b u_b^b)) = 0 \). Therefore,
\[ \nabla \cdot u_b^b = 0 . \tag{44} \]
Equation (44) indicates that the first order time average flow in the boundary layer is incompressible. The momentum equation within the boundary layer in the order of \( \epsilon \) after taking the time average. The momentum equation in the radial direction, \( \eta \) is,
\[ \frac{\partial \rho_b^b}{\partial \eta} = 0 \tag{45} \]
The momentum equation in the tangential direction, \( \theta \), is
\[ \frac{1}{2} \frac{\partial^2 \rho_b^b}{\partial \theta^2} + \frac{1}{r} \frac{\partial}{\partial \theta} \left( \frac{(ka)^2 \rho_b^b u_{b0}}{\partial \theta} \right) + \left( u_{b0} \frac{\partial u_{b0}}{\partial \eta} \right) - \frac{\sqrt{1 - \bar{\mu}^2}}{R} u_{b0} \frac{\partial u_{b0}}{\partial \mu} + \frac{\sqrt{2}}{R|M|} u_{b0} u_{b0} \right) . \tag{46} \]
Using equations (40) and (41) in (46) and taking the derivative of (46) with respect to \( \eta \) while keeping in mind (45), the equation for \( u_{b1} \) to \( O(ka) \) becomes,
\[ \frac{\partial^3 u_{b1}}{\partial \eta^3} = \frac{10}{3} \epsilon^\prime ka \frac{\bar{\mu}}{M} \left[ \sin \eta [(1 - i) \cos \phi - (1 + i) \sin \phi] \right. \]
\[ \left. + \cos \eta [(1 + i) \sin \phi + (1 - i) \sin \phi] + \sqrt{2} M \sin \eta [(1 + i) \sin \phi - (1 + i) \cos \phi] + \cos \eta [(1 - i)(\sin \phi + \cos \phi)) \right) \]
\[ + |M|^2(\sin \eta [(1 + i) \sin \phi + (1 - i) \cos \phi]) \]
\[ + \cos \eta [(1 - i) \sin \phi - (1 + i) \cos \phi]) \right\} \right] . \tag{47} \]
After successive integration of the above equation and taking the limit of \( u_{b1} = o(\eta) \) as \( \eta \rightarrow \infty \),
\[ u_{b1} = 5 \epsilon^\prime ka \frac{\bar{\mu}}{M} \left[ \sin \eta [\cos \phi - i \sin \phi] \right. \]
\[ \left. + \cos \eta [\sin \phi + \cos \phi] + \sqrt{2} M \left( \cos \eta [(1 - i) \sin \phi + \cos \phi] - \sin \eta (i \cos \phi) \right) \right) \]
\[ + |M|^2(\sin \eta [(1 + i) \sin \phi - \cos \phi] - \cos \eta [(1 + i) \sin \phi + i \cos \phi]) + C_2 \eta + C_3 \right] . \tag{48} \]
where \( C_2 \) and \( C_3 \) are constants of integration to be determined from the boundary conditions and matching with the far field. Terms in the order of \( O(ka)^2 \) are neglected (see e.g., Rednikov et al. 2006). Before we proceed, we express the velocity as a stream function, i.e.,
\[ u_{b1} = -\frac{|M|}{\sqrt{2} R \sqrt{1 - \bar{\mu}^2}} \frac{\partial \psi_b}{\partial \eta}, \tag{49} \]
and,
\[ u_b = -\frac{1}{R^2} \frac{\partial \psi_b}{\partial \mu} . \tag{50} \]
The far field and the boundary condition for the stream function is
\[ \psi_1^b = o(\eta^2) \] (51)
in order to make the far field streaming velocity vanish as \( \eta \to \infty \). Form the free surface condition at the interface of the bubble and the fluid around it,
\[ \frac{\partial^2 \psi_1^b}{\partial \eta^2} \bigg|_{\eta=0} = \frac{\sqrt{2} R[M]}{|M|^2} \frac{\partial \psi_1}{\partial \eta} \bigg|_{\eta=0} \] (52)
and the zero normal velocity at the first order,
\[ \frac{\partial \psi_1^b}{\partial \mu} \bigg|_{\eta=0} = 0 \] (53)
After changing the velocity to the stream function, keeping only the real part, and invoking the boundary conditions above,
\[ \psi_1^b = -\frac{\sqrt{2}}{6} \frac{e^{\prime} k a}{|M|} \frac{\bar{\mu}(1 - \bar{\mu}^2)}{|M|^2 + (|M| + \sqrt{2}|M|)^2} \left\{ e^{-\eta} \left[ \sin \eta \left( \sin \phi - \cos \eta \left( \sin \phi + 2 \cos \phi \right) \right) \right. \\
+ \sqrt{2}|M| \left( \sin \eta \left( \cos \phi - \sin \eta \right) \right) \left( \sin \phi + \cos \phi \right) + |M|^2 \left( \sin \eta \left( \cos \phi - \sin \eta \right) \right) \left. \left( \sin \phi + \cos \phi \right) \right] \\
- \eta \left[ \sin \phi + 2 \cos \phi + 2\sqrt{2}|M| \left( \sin \phi + \cos \phi \right) \right] + \sin \phi + 2 \cos \phi + \sqrt{2}|M| \left( \sin \phi + \cos \phi \right) \\
- |M|^2 \left( \cos \phi - \sin \phi \right) \right] \\
- \eta \frac{5}{6} e^{\prime} k a \frac{\bar{\mu}(1 - \bar{\mu}^2)}{|M|^2 + (|M| + \sqrt{2}|M|)^2} \left[ -2 \sin \phi - 2\sqrt{2}|M| \left( \sin \phi + \cos \phi \right) + 2|M|^2 \left( \sin \phi - \cos \phi \right) \right] \\
- C_2 \sqrt{1 - \bar{\mu}^2} \left( \eta R^2 + \eta^2 \frac{\sqrt{2} R}{|M|} \right) . \] (54)
To find the constant \( C_2 \) in equation (54) we match with the streaming from the flow outside the boundary layer. The flow outside the boundary layer is considered to be incompressible. Therefore, the stream function, \( \psi_1 \), can be introduced (Zhao et al. 1999), resulting in
\[ D^4 \psi_1 = 0 , \] (55)
where \( D^2 = \frac{\partial^2}{\partial r^2} + \frac{(1-\bar{\mu}^2)}{r^2} \frac{\partial^2}{\partial \varphi^2} \)
Equation (55) indicates that the flow outside the boundary layer is Stokes flow. Taking into account the matching requirement of the behavior of the standing wave and knowing the condition \( \psi_1 = o(r^2) \) as \( r \to \infty \), the general solution is found to be
\[ \psi_1 = \left( b_2 + \frac{D_2}{r^2 - \frac{3b_4}{r^4}} \right) \bar{\mu}(1 - \bar{\mu}^2) + \left( \frac{7b_4}{r^2} + \frac{7D_4}{r^4} \right) \bar{\mu}^3(1 - \bar{\mu}^2) . \] (56)
By introducing the \( \eta \) variable in equation (56) and letting \( \eta \to \infty \), then matching it with equation (54) while letting \( \eta = 0 \) gives the unknown constants. The solutions for \( \psi_1 \) and \( \psi_1^b \) turn out to be
\[ \psi_1 = \frac{5}{6} e^{\prime} k a \frac{\bar{\mu}(1 - \bar{\mu}^2)}{|M|^2 + (|M| + \sqrt{2}|M|)^2} \left\{ \left( 1 - \frac{1}{r^2} \right) \left( \sin \phi + \cos \phi + \sqrt{2}|M| \left( \sin \phi + \cos \phi \right) \right) \\
+ \frac{|M|^2}{2} \left( \sin \phi + \sqrt{2}|M| \left( \sin \phi + \cos \phi \right) - |M|^2 \left( \sin \phi - \cos \phi \right) \right) \right\} \] (57)
and,
\[ \psi_1^b = -\frac{\sqrt{2}}{6} \frac{e^{\prime} k a}{|M|} \frac{\bar{\mu}(1 - \bar{\mu}^2)}{|M|^2 + (|M| + \sqrt{2}|M|)^2} \left\{ \left( \sin \phi + 2 \cos \phi + \sqrt{2}|M| \left( \sin \phi + \cos \phi \right) - |M|^2 \left( \cos \phi - \sin \phi \right) \right) \\
+ e^{-\eta} \sin \eta \left( \sin \phi - \cos \eta \left( \sin \phi + 2 \cos \phi \right) \right) \left( \sin \eta - \cos \eta \right) \left( \sin \phi + \cos \phi \right) \\
+ |M|^2 \left( \cos \phi - \sin \phi \right) \left( \cos \phi - \sin \phi \right) \right] \\
- \eta \left[ \sin \phi + 2 \cos \phi + \sqrt{2}|M| \left( \sin \phi + \cos \phi \right) + 2|M|^2 \left( \sin \phi + \cos \phi \right) \right] \\
- \sqrt{2}|M|^2 \left( \sin \phi - \cos \phi \right) \right\} . \] (58)
The existence of $\psi_1$ indicates that the steady streaming to $O(\varepsilon)$ occurs outside the boundary layer as well. Equations (57) and (58) can further be simplified by realizing that the frequency parameter was assumed to be large, i.e., $|M|^2 \gg 1$. After discarding terms of $O(|M|^{-1})$ and smaller, we obtain the streaming functions

$$\psi_1 = \frac{5}{24} \varepsilon' k a \tilde{\mu}(1 - \tilde{\mu}^2) \left(1 - \frac{1}{r^2}\right) \left(|M|^2 - 1\right) \cos \phi - \left(|M|^2 - 2\sqrt{2}|M| + 2\right) \sin \phi$$

(59)

and,

$$\psi_1^b = -\frac{\sqrt{2}}{|M|} \frac{5}{12} \varepsilon' k a \tilde{\mu}(1 - \tilde{\mu}^2) \left(\sin \phi - \cos \phi + e^{-\eta}(\sin \eta + \cos \eta)(\sin \phi + \cos \phi)ight)$$

$$-\eta \left[4 \sin \phi - \sqrt{2}|M| (\sin \phi - \cos \phi)\right].$$

(60)

Equation (59) can be further simplified by letting $\tan \alpha = \frac{|M|^2 - 1}{|M|^2 - 2\sqrt{2}|M| + 2}$, which gives

$$\psi_1 = \frac{5}{24} \varepsilon' k a \tilde{\mu}(1 - \tilde{\mu}^2) \frac{(|M|^2 - 2\sqrt{2}|M| + 2)}{\cos \alpha} \left(1 - \frac{1}{r^2}\right) \sin(\alpha - \phi)$$

(61)

where the sine function in equation (61) gives the value of the phase angle which corresponds to the shifting of the flow direction.

3- Discussion:

From equations (57), (58), (59), and (60) steady streaming is found to exist for both the outer and the inner flow fields. As can be seen from Figure 2(a), which is a graph of the outer region streaming, the flow has a symmetrical pattern about the equatorial plane of the bubble. The difference is in the direction of the flow between the upper and the lower plane. For a value of $|M| = 100$ and $-134^\circ < \phi < 46^\circ$, the streaming flow direction is from the equatorial plane to the poles. Outside this range, i.e., $46^\circ < \phi < 180^\circ$ and $-180^\circ < \phi < -134^\circ$, the flow is from the poles to the equatorial plane. The reversal takes place at $\phi = 46^\circ$ and $\phi = -134^\circ$, at which point the streaming flow to this order vanishes. Of course, at higher order there could be some flow and further investigation is needed. It can also be noticed that the streamlines are open and merge with the main flow and there is no closed circulation. The inner region (boundary layer) streaming shown in Figure 2(b), has the same behavior with respect to the phase angle as the streaming of the outer region. It should be noted that the large $|M|$ approximation in the current analysis significantly restricts the validity to small velocity amplitude so that the perturbed field to $O(\varepsilon)$ remain indeed small.

FIGURE 2. (a) The outer streamlines of the steady streaming around a bubble located in the velocity node of a standing wave for phase angle of 90°. (b) Streamlines in the boundary layer at the bubble surface. The image is stretched in the radial direction and the phase angle is 90°.
REFERENCES