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3aSP2.  Phase-space equation for wave equations

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Transforming a space-time function into the phase space of position and wave number offers considerable insight into the nature of the function, and also has many practical applications. If the function is governed by a wave equation then the common procedure is to solve the wave equation and then calculate the phase space distribution function for the solution. We show that significant advantages ensue if one transforms the original differential equation into a phase space differential equation. We give a number of examples and show that phase space equations are often more revealing than the original equation and lead to new approximation methods.

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INTRODUCTION

In 1932 Wigner transformed the Schrödinger equation into phase space, that is, into a partial differential equation that involves position and momentum [7]. Since that time it has been shown that there are many advantages to the phase space equation both from the point of view of insight and practical calculation. In a series of papers Galleani and Cohen derived methods to transform arbitrary linear ordinary equations and partial differential equations into phase space [1-3]. In this paper we discuss the applications of these methods to wave propagation with dispersion and attenuation. Also, Loughlin and Cohen have shown that the Wigner distribution of a propagating wave presents advantages and in particular allows one to obtain new approximations for pulse propagation [5,6]. Here, we show how the differential equation method leads to the same approximation.

We consider a general wave equation with constant coefficients

\[ \sum_{n=0}^{N} a_n \frac{\partial^n u}{\partial t^n} = \sum_{n=0}^{M} b_n \frac{\partial^n u}{\partial x^n} \]  

(1)

The standard method for solving it is to substitute \( e^{ikx - i\omega t} \) into Eq. (1) to obtain an algebraic equation

\[ \sum_{n=0}^{N} a_n (i\omega)^n = \sum_{n=0}^{M} b_n (i k)^n \]  

(2)

One then solves for \( \omega \) as a function of \( k \)

\[ \omega = \omega(k) \]  

(3)

Generally there is more than one solution to Eq. (2), each solution is called a mode. If we know \( u(x, t') \) at time \( t' \) then the solution at a later time for a particular mode is [4]

\[ u(x, t) = \frac{1}{\sqrt{2\pi}} \int S(k, t') e^{ikx - i\omega(k)(t-t')} \, dk \]  

(4)

where \( S(k, t') \) is obtained from the initial pulse, \( u(x, t') \), by way of

\[ S(k, t') = \frac{1}{\sqrt{2\pi}} \int u(x, t') e^{-ikx} \, dx \]  

(5)

Defining

\[ S(k, t) = S(k, t') e^{-i\omega(k)(t-t')} \]  

(6)

we have

\[ u(x, t) = \frac{1}{\sqrt{2\pi}} \int S(k, t) e^{ikx} \, dk \]  

(7)

\[ S(k, t) = \frac{1}{\sqrt{2\pi}} \int u(x, t) e^{-ikx} \, dx \]  

(8)

shows that \( u(x, t) \) and \( S(k, t) \) form Fourier transform pairs for all \( t \).

Generally \( \omega(k) \) is complex and we write

\[ \omega(k) = \omega_R(k) + i\omega_I(k) \]  

(9)

If \( \omega_I(k) = 0 \) we have no damping, otherwise we have propagation with damping.

We now obtain the differential equation for a mode. In Eq. (4) differentiate with respect to time and multiply by \( i \) to obtain

\[ i \frac{\partial}{\partial t} u(x, t) = \frac{1}{\sqrt{2\pi}} \int \omega(k) S(k, 0) e^{-i\omega(k)t} e^{ikx} \, dk \]  

(10)

\[ = \frac{1}{\sqrt{2\pi}} \int \omega \left( \frac{1}{i} \frac{\partial}{\partial x} \right) S(k, 0) e^{-i\omega(k)t} e^{ikx} \, dk \]  

(11)

\[ = \frac{1}{\sqrt{2\pi}} \omega \left( \frac{1}{i} \frac{\partial}{\partial x} \right) \int S(k, 0) e^{-i\omega(k)t} e^{ikx} \, dk \]  

(12)
We now expand and therefore

\[ i \frac{\partial}{\partial t} u(x, t) = \omega \left( \frac{1}{i} \frac{\partial}{\partial x} \right) u(x, t) \]  

(13)

Also, the differential equation for the spatial spectrum, \( S(k, t) \), is

\[ i \frac{\partial}{\partial t} S(k, t) = \omega(k) S(k, t) \]  

(14)

**Greens function.** If we combine Eq. (5) with Eq. (4) we have

\[ u(x, t) = \frac{1}{2\pi} \int u(x', t') e^{-ikx'} e^{ikx - i\omega(k)(t-t_0)} \, dk \, dx' \]  

(15)

and therefore one can write

\[ u(x, t) = \int u(x', t') G(x, x'; t, t') \, dx' \]  

(16)

where \( G(x, x'; t, t') \) is the Greens function and given by

\[ G(x, x'; t, t') = \frac{1}{2\pi} \int u(x', t_0) e^{ik(x-x') - i\omega(k)(t-t')} \, dk \]  

(17)

**Wigner distribution**

The joint position - wavenumber Wigner distribution for the wave \( u(x, t) \) is

\[ W(x, k; t) = \frac{1}{2\pi} \int u^*(x - \frac{1}{2} \tau, t) u(x + \frac{1}{2} \tau, t) e^{-itk} \, d\tau \]  

(18)

which in terms of the spatial spectrum is given by

\[ W(x, k; t) = \frac{1}{2\pi} \int S^*(k + \frac{1}{2} \theta, t) S(k - \frac{1}{2} \theta, t) e^{-i\theta k} \, d\theta \]  

(19)

Using the method described in [1-3] one can derive the Wigner distribution corresponding to Eq. (13)

\[ i \frac{\partial}{\partial t} W(x, k, t) = \left[ \omega \left( k + \frac{1}{2i} \frac{\partial}{\partial x} \right) - \omega^* \left( k - \frac{1}{2i} \frac{\partial}{\partial x} \right) \right] W(x, k, t) \]  

(20)

Explicitly

\[ i \frac{\partial}{\partial t} W(x, k, t) = \left[ \omega_R \left( k + \frac{1}{2i} \frac{\partial}{\partial x} \right) - \omega_R \left( k - \frac{1}{2i} \frac{\partial}{\partial x} \right) \right] W(x, k, t) \]  

(21)

\[ + i \left[ \omega_I \left( k + \frac{1}{2i} \frac{\partial}{\partial x} \right) + \omega_I \left( k - \frac{1}{2i} \frac{\partial}{\partial x} \right) \right] W(x, k, t) \]  

(22)

We now expand \( \omega_R \) and \( \omega_I \) in a Taylor series,

\[ \frac{\partial}{\partial t} W(x, k, t) = \sum_{n=0}^{\infty} \frac{\omega_R^{(2n)}(k)}{(2n+1)!} \frac{\partial^{2n+1}}{\partial x^{2n+1}} W(x, k, t) + \sum_{n=0}^{\infty} \frac{\omega_I^{(2n)}(k)}{(2n)!} \frac{\partial^{2n-1}}{\partial x^{2n-1}} \left( \frac{\partial}{\partial x} \right)^{2n} \]  

(23)

\[ \sim 2\omega_I(k) - \omega_R^{(1)}(k) \frac{\partial}{\partial x} - 1 \frac{\omega_R^{(2)}(k)}{4} \frac{\partial^2}{\partial x^2} + \frac{1}{24} \omega_R^{(3)}(k) \left( \frac{\partial}{\partial x} \right)^3 \]  

(24)

Suppose we just keep terms up to the first derivative and call the approximate solution \( W_a \), then

\[ \frac{\partial}{\partial t} W_a(x, k, t) \sim 2\omega_I(k) - v_g(k) \frac{\partial}{\partial x} W_a(x, k, t) \]  

(25)
The Greens function for the Wigner distribution. The relation between \( W(x, k, t) \) and \( W(x', k, t) \) is

\[
W(x, k, t) = \frac{1}{2\pi} \int W(x', k, t') e^{-i\theta(x-x')} e^{i[\omega(t-k/2) - \omega(k-0/2)(t-t')]} d\theta dx'
\]

This relates the Wigner distribution at time \( t \) with the initial Wigner distribution. One can write a Greens function for the Wigner distribution. Defining \( G_W \) by

\[
W(x, k, t) = \frac{1}{2\pi} \int W(x', k', t') G_W(x, k, t; x', k', t') dx' dk'
\]

we have that

\[
G_W(x, k, t; x', k', t') = \frac{1}{2\pi} \delta(k - k') \int e^{i\theta(x-x')} e^{i[\omega(t-k/2) - \omega(k-0/2)(t-t')]} d\theta
\]

Relation between Damped and Undamped Waves

We now consider the relationship between the evolution of a wave with and without damping. We call the solution to the two cases \( u(x, t) \) and \( f(x, t) \) respectively where

\[
u(x, t) = \frac{1}{\sqrt{2\pi}} \int S(k, 0) e^{i(kx - i\omega t)k} dk \quad \text{with damping}
\]

\[
f(x, t) = \frac{1}{\sqrt{2\pi}} \int S(k, 0) e^{i(kx - i\omega t)k} dk \quad \text{without damping}
\]

For both cases we take the initial pulse to be the same

\[
u(x, 0) = f(x, 0)
\]

In the above we have taken \( t' = 0 \). We now derive an explicit relation between \( u(x, t) \) and \( f(x, t) \). The corresponding spatial spectra are

\[
S_u(k, t) = S(k, 0) e^{-i\omega t} e^{i(kx - i\omega t)k}
\]

\[
S_f(k, t) = S(k, 0) e^{-i\omega t} k
\]

and therefore

\[
S_u(k, t) = S_f(k, t) e^{i\omega t} k
\]

By taking the Fourier transform of both sides one obtains

\[
u(x, t) = \frac{1}{\sqrt{2\pi}} \int S_f(k, t) e^{i\omega t} e^{i(kx - i\omega t)k} dk
\]

and substituting for \( S_f(k, t) \) we have

\[
u(x, t) = \frac{1}{2\pi} \int f(x', t) e^{-i(k(x' - x))} e^{i\omega t} dx' dk
\]

This explicitly shows how one can obtain \( u(x, t) \) from \( f(x, t) \). We write Eq. (38) as

\[
u(x, t) = \int f(x', t) L(x' - x, t) dx'
\]
where

\[
L(x' - x, t) = \frac{1}{2\pi} \int e^{-ik(x' - x)} e^{i\omega t} dk
\]  \hspace{1cm} (40)

Similarly,

\[
f(x, t) = \frac{1}{\sqrt{2\pi}} \int S_u(k, t) e^{-i\omega t} e^{ikx} dk
\]  \hspace{1cm} (41)

\[
= \frac{1}{2\pi} \int u(x', t) e^{-i\omega t} e^{ikx} dk dx'
\]  \hspace{1cm} (42)

and hence

\[
f(x, t) = \int u(x', t) L(x' - x, -t) dx'
\]  \hspace{1cm} (43)

We now examine this from the differential equation point of view. We write

\[
i \frac{\partial}{\partial t} u(x, t) = \left[ \omega_R \left( \frac{1}{i} \partial_x \right) + i \omega_I \left( \frac{1}{i} \partial_x \right) \right] u(x, t) \quad \text{with damping}
\]  \hspace{1cm} (44)

\[
i \frac{\partial}{\partial t} f(x, t) = \omega_R \left( \frac{1}{i} \partial_x \right) f(x, t) \quad \text{without damping}
\]  \hspace{1cm} (45)

We now show the relation of Eq. (39) to these differential equations. Differentiating Eq. (39) with respect to time and multiplying by \(i\), we have

\[
i \frac{\partial}{\partial t} u(x, t) = \int i \left[ \frac{\partial}{\partial t} f(x', t) L(x' - x, t) + f(x', t) \frac{\partial}{\partial t} L(x' - x, t) \right] dx'
\]  \hspace{1cm} (46)

\[
= \int \left( L(x' - x, t) \omega_R \left( \frac{1}{i} \partial_x \right) f(x', t) + i f(x', t) \frac{\partial}{\partial t} L(x' - x, t) \right) dx'
\]  \hspace{1cm} (47)

Now consider

\[
\frac{\partial}{\partial t} L(x' - x, t) = \frac{1}{2\pi} \frac{\partial}{\partial t} \int e^{-ik(x' - x)} e^{i\omega t} dk
\]  \hspace{1cm} (48)

\[
= \frac{1}{2\pi} \int \omega_I(k) e^{-ik(x' - x)} e^{i\omega t} dk
\]  \hspace{1cm} (49)

\[
= \omega_I \left( \frac{1}{i} \partial_x \right) L(x' - x, t)
\]  \hspace{1cm} (50)

Also consider

\[
\int L(x' - x, t) \omega_R \left( \frac{1}{i} \partial_x \right) f(x', t) dx' = \int f(x', t) \omega_R \left( -\frac{1}{i} \partial_x \right) L(x' - x, t) dx'
\]  \hspace{1cm} (51)

\[
= \int f(x', t) \omega_R \left( \frac{1}{i} \partial_x \right) L(x' - x, t) dx'
\]  \hspace{1cm} (52)

and therefore

\[
i \frac{\partial}{\partial t} u(x, t) = \int \left[ f(x', t) \omega_R \left( \frac{1}{i} \partial_x \right) L(x' - x, t) + i f(x', t) \omega_I \left( \frac{1}{i} \partial_x \right) L(x' - x, t) \right] dx'
\]  \hspace{1cm} (53)

\[
i \frac{\partial}{\partial t} u(x, t) = \left[ \omega_R \left( \frac{1}{i} \partial_x \right) + i \omega_I \left( \frac{1}{i} \partial_x \right) \right] \int f(x', t) L(x' - x, t) dx'
\]  \hspace{1cm} (54)

\[
= \left[ \omega_R \left( \frac{1}{i} \partial_x \right) + i \omega_I \left( \frac{1}{i} \partial_x \right) \right] u(x, t)
\]  \hspace{1cm} (55)

which is Eq. (44)
CONCLUSION

We point out that just as we related the propagation of a pulse when there is attenuation with the pulse without attenuation one can also do so for the Wigner distribution. The result is

\[ W(x,k,t) = \frac{1}{2\pi} \int W_f(x',k,t) e^{i\omega_I(k+\theta/2) - i\omega_I(k-\theta/2)} t e^{-i\theta(x-x')} d\theta dx' \]  

(56)

As an example suppose

\[ W_f(x,k,t) = W(x-v_g(k)t), k, 0) \]  

(57)

Then, with attenuation we have

\[ W(x,k,t) = \frac{1}{2\pi} \int W_f(x'-v_g(k)t), k, 0) e^{i\omega_I(k+\theta/2) - i\omega_I(k-\theta/2)} t e^{-i\theta(x-x')} d\theta dx' \]  

(58)

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